**Problem 1** (5 points). *Show the following assertions about elements of an ordered field $(K, \leq)$:*

- *$x$ is positive if and only if $-x$ is negative.*
- *If $xy \neq 0$ then $xy$ is positive if and only if $x$ and $y$ have the same sign, i.e., $x$ and $y$ are either both positive or both negative.*
- *If $x \geq 0$ and $y \geq 0$ then $x + y \geq 0$, and equality holds if and only if $x = y = 0$.*

In what follows, let $K$ be any field. If $A$ is a finite dimensional $K$-algebra and $a \in A$, let $\mathrm{Tr}_{A/K}(a)$ and $\mathrm{N}_{A/K}(a)$ be the trace and the determinant of the $K$-linear endomorphism $x \to ax$ of the $K$-vector space $A$. In particular, we will apply these definitions when $A$ is a finite field extension of $K$. By the properties of trace and determinant of linear operators well-known from linear algebra, we have

**Fact 1.** *For $a, b \in A$, $\mathrm{Tr}(a+b) = \mathrm{Tr}(a) + \mathrm{Tr}(b)$ und $\mathrm{N}(ab) = \mathrm{N}(a)\mathrm{N}(b)$.*

Also,

**Problem 2** (2 points). *If $B$ is another finite-dimensional $K$-algebra. Let $A \oplus B$ we equipped with the product $(a, b)(\alpha, \beta) = (a\alpha, b\beta)$, and let $K \to A \oplus B$ send $k$ to $(k, k)$. If $a \in A$ and $b \in B$, then*

$$\mathrm{Tr}_{A \oplus B/K}(a, b) = \mathrm{Tr}_{A/K}(a) + \mathrm{Tr}_{B/K}(b)$$
$$\mathrm{N}_{A \oplus B/K}(a, b) = \mathrm{N}_{A/K}(a)\mathrm{N}_{B/K}(b)$$

**Problem 3** (2 points). *Let $L/K$ be a finite field extension, let $\otimes$ always denote the tensor product over $K$, and equip $A \otimes L$ with the product $(a \otimes l)(\alpha \otimes \lambda) = (a\alpha) \otimes (l\lambda)$ and the ring morphism from $L$ sending $l$ to $1 \otimes l$. Then $\mathrm{Tr}_{A \otimes L/L}(a \otimes 1) = \mathrm{Tr}_{A/K}(a) \otimes 1$ and $\mathrm{N}_{A \otimes L/L}(a \otimes 1) = \mathrm{N}_{A/K}(a) \otimes 1$.*

For the following two problems, let $L/K$ be a finite field extension, $V$ a finite dimensional $L$-vector space and $A$ an $L$-linear endomorphism of $V$. If $M$ is $K$ or $L$, let $\det_M(A)$ and $\mathrm{Tr}_M(A)$ be the determinant trace of $A$, viewed as an endomorphism of the finite dimensional $K$-vector space $V$.

**Problem 4** (5 points). *Then $\det_K(A) = \mathrm{N}_{L/K}\det_L(A)$.*

**Problem 5** (2 points). *In the same situation. we have $\mathrm{Tr}_K(A) = \mathrm{Tr}_{L/K}\mathrm{Tr}_L(A)$.*

**Remark 1.** *The second of the above two problems should be rather straightforward. The first one may be a bit harder. For instance, one can use results from linear algebra to reduce to the case where $A$ has a simple matrix representation, and this case can be dealt with by a straightforward calculation.*

**Problem 6** (1 point)**.** *If $M/L$ is another finite field extension, then*

$$\mathrm{Tr}_{M/K}(m) = \mathrm{Tr}_{L/K}\mathrm{Tr}_{M/L}(m)$$
$$\mathrm{N}_{M/K}(m) = \mathrm{N}_{L/K}\mathrm{N}_{M/L}(m)$$

*for all $m \in M$.*

**Problem 7** (4 points)**.** *If $K$ is a field of odd characteristic, then every element of $K$ is a sum of squares in $K$.*

**Remark 2.** *This can be derived as a special case of a result on field orderings which will be shown next week. However, this derivation will not be accepted here as the result has a short and relatively straightforward proof using the binomial theorem. The fact that the set of sums of squares in $K$ is multiplicatively closed may be used without a proof.*

Solutions should be submitted in the lecture on Friday, October 20.