François, K., Löwe, B., Müller, T., Van Kerkhove, B., editors,
Foundations of the Formal Sciences VII
Bringing together Philosophy and Sociology of Science

Albert Lautman: Dialectics in mathematics

BRENDAN LARVOR[*]

Department of Philosophy, University of Hertfordshire, Hatfield, Hertfordshire AL10 9AB, United Kingdom

E-mail: `b.p.larvor@herts.ac.uk`

Albert Lautman (1908–1944) is a rare example of a twentieth-century philosopher whose engagement with contemporary mathematics goes beyond the 'foundational' areas of mathematical logic and set theory. He insists that (what were in his day) the new mathematics of topology, abstract algebra, class field theory and analytic number theory have a philosophical significance that distinguishes them from the mathematics of earlier eras. Specifically, these new areas of mathematics reveal underlying dialectical structures not found in earlier mathematics. In a series of short papers and two longer theses (*Essay on the unity of the mathematical sciences in their current development* and *Essay on the notions of structure and existence in mathematics*)[1], Lautman argues this claim from a philosophical perspective rooted in certain of the later dialogues of Plato. However, Lautman was not satisfied with Plato's conception of the relation between dialectical Ideas and the matter in which they are realised. In one of his last papers, *New research on the dialectical structure of mathematics*[2], Lautman bolsters his Platonism with an appeal to Heidegger's 'ontological' distinction between phenomenology and science.[3] We may therefore regard this paper as the most advanced expression available of Lautman's philosophy of mathematics.

---

[*]A French version of this paper was published as *Albert Lautman, ou la dialectique dans les mathématiques* in the journal *Philosophiques* 37(1):75–94, 2010. I am grateful to David Corfield, Nicholas Joll and Danièle Moyal-Sharrock for their careful reading of earlier versions of this paper.

[1]Henceforth, page numbers refer to the 2006 Vrin edition of Lautman's complete works, (Lautman, 2006).

[2](Lautman, 2006, pp. 235–257); this paper was first published in 1939 in a series edited by Jean Cavaillès and Raymond Aron.

[3]As expressed in Heidegger's 1928 lecture *Vom Wesen des Grundes*. Quotations here are from McNeill's 1998 translation *On the Essence of Ground*. Lautman quotes Corbin's 1938 French translation.

Received by the editors: 17 December 2008.
Accepted for publication: 25 March 2010.

In this paper, I shall first explore Lautman's conception of dialectics by a consideration of his references to Plato and Heidegger. I shall then compare the dialectical structures that he found in contemporary mathematics with the model that emerges from his philosophical sources. I shall argue that the structures that he discovered in mathematics are richer than his Platonist model suggests, and that Heidegger's 'ontological' distinction is less useful than Lautman seemed to believe.

1 Plato

In his major case studies, Lautman developed a picture of modern mathematics (that is, mathematics in the early twentieth century) as the expression or realisation of fundamental conceptual oppositions (such as continuous/discontinuous, global/local, finite/infinite, symmetric/anti-symmetric).[4] He referred to the opposing terms as *notions*; dialectical *Ideas* envisage possible relations between such pairs of dialectical notions (Lautman, 2006, pp. 242–243). This terminology is a conscious reference to Plato, and he is careful to distinguish his appeal to Plato from 'Platonism' as philosophers of mathematics usually use the term. In philosophy of mathematics, 'Platonism' usually denotes the view that mathematical objects exist independently of the thought and talk of mathematicians. Lautman insisted that this was a misreading of Plato (Lautman, 2006, p. 230); in any case, this kind of 'Platonism' is not Lautman's view.[5]

Lautman never quotes Plato directly, and he mentions just three Platonic texts: *Philebus*, the *Sophist* (twice), and *Timaeus* (twice). Scholars usually count these among the 'later' dialogues of Plato (though the *Sophist* is continuous with the *Theaetetus* and implicitly refers to the *Parmenides*—both middle period dialogues). What matters for our purpose is that Plato's *theory of forms* is largely absent from his later works. The *Ideas* in the later dialogues are not blueprints for material objects. Similarly, Lautman's mathematical Platonism was not a 'copy-theory'. As he points out, we might think of material reality as inchoate matter somehow shaped into material copies of non-material 'forms', but this model cannot apply to the relation between mathematical theories and the dialectical ideas that (in Lautman's term) *dominate* them (Lautman, 2006, p. 238).

[4]This list is drawn from the two long essays. In *New Research on the dialectical structure of mathematics* he offers a slightly different list of dialectical pairs, "wholes and parts, situational and intrinsic properties, basic domains and objects defined on these domains, formal systems and their models, etc.." (Lautman, 2006, p. 243)

[5]"Dans le débat ouvert entre formalistes et intuitionnistes, [...] les mathématiciens ont pris l'habitude de désigner sommairement sous le nom de platonisme toute philosophie pour laquelle l'existence d'un être mathématique est tenue pour assuré [...] c'est là une connaissance superficielle du platonisme [...]" (Lautman, 2006, p. 230).

1.1 The *Sophist*

In a short paper of 1937 called *L'axiomatique et la méthode de division*[6], Lautman refers to *Philebus* and the *Sophist* together:

> The movement from so-called 'elementary' notions to abstract notions does not [...] appear as the subsumption of the particular under the general, but rather as the division or analysis of a 'mixture' which tends to yield simple notions in which this mixture participates. It is, therefore, not the Aristotelian logic of genus and species at work here, but the Platonic method of division, as taught in the *Sophist* and *Philebus*, in which the unity of Being is a unity of composition and a starting-point in the search for principles that are unified in Ideas.[7]

The *Sophist* is a discussion between a young man, Theaetetus, and a stranger from Elea, "a comrade of the circle of Parmenides and Zeno, and a man very much a philosopher" (216A). The initial question is whether the words 'sophist', 'statesman' and 'philosopher' name one, two or three types of thing, and what that thing is or those things are. The nameless stranger asks for an "interlocutor [who] submits to guidance easily" (217D); Socrates proposes young Theaetetus. Thus, Plato allows the unnamed philosopher to develop his position at length without having to fend off a Socratic interrogation (this is a feature of Plato's later works; in the eponymous dialogue, Timaeus has the floor to himself after the preliminary civilities). Thereafter, Socrates vanishes from the text, so we do not have the luxury of inferring Plato's view from Socrates's words.

The Eleatic philosopher proceeds by division, that is, by making one distinction after another. He illustrates this technique with the term 'angler'. He first distinguishes gathering arts from manufacturing arts; then the gathering arts are divided into trading and 'mastering' or getting the better of; getting the better of divides into competition and hunting; hunting divides according to quarry (animal or other); animals swim or walk; swimming animals divide into water-fowl and fish; fishing divides into trapping (with nets, traps, etc.) and striking; striking divides into striking down with a trident and up with a hook. The resulting tree of categories is his account of 'angler'. He then proceeds to apply the same technique to the term

[6] *Axiomatics and the method of division*; (Lautman, 2006, pp. 69–80).

[7] "Le passage des notions dites 'élémentaires' aux notions abstraits ne se présente donc pas comme une subsomption du particulier sous le général mais comme la division ou l'analyse d'un 'mixte' qui tend à dégager les notions simples auxquelles ce mixte participe. Ce n'est donc pas la logique aristotélicienne, celle des genres et des espèces qui intervient ici, mais la méthode platonicienne de division, telle que l'enseignent le Sophiste et le Philèbe pour laquelle l'unité de l'Être est une unité de composition et un point de départ vers la recherche des principes qui s'unissent dans les Idées" (Lautman, 2006, pp. 78–79).

'sophist', and this discussion occupies the remainder of the dialogue. The Eleatic philosopher develops several different accounts of 'sophist' (231D-E), which leads to a methodological discussion, including a debate about the possibility of numbering non-beings (238B). The discussion refers to itself, because Theodorus introduced the Eleatic stranger as a philosopher, presumably in virtue of his logical technique (253C).[8] But if the method of division turns out to be merely a spurious word-game, then perhaps *he* is a sophist. Certainly, his choices of divided categories seem arbitrary. For example, he might have divided fishing according to whether or not bait is used, in which case trident-fishing and net-fishing would have been divided from angling and the use of baited traps. Young Theaetetus submits to the philosopher's guidance rather too easily, and certainly more easily than Socrates would have done.

Whatever Plato's intent in giving an unnamed, generic Eleatic philosopher an easy ride, Lautman takes the method of division as an unproblematic technique, and makes no mention of its proper companion, the 'method of collection'. In the text immediately before the excerpt quoted above, Lautman runs through a list of mixtures, that is, mathematical items that 'participate' in two heterogeneous categories. Namely: arithmetical equality is the only equivalence relation such that the number of equivalence classes equals the cardinality of the base domain; the idea of multiplication refers both to the creation of arithmetical products and to the idea of operators on a domain; unity can be thought of either as the unit element of a ring of numbers or as the identity element in a domain of operators; the length of a segment depends on the size of the segment but at the same time depends on a convention; absolute value in classical algebra includes the notion of ordering but also the notion of the completeness of a field. He goes on to claim that some of these mixtures (arithmetical equality; multiplication; absolute value) are examples of the dialectical relation between the intrinsic and relational properties of mathematical objects (Lautman, 2006, pp. 78–79). He then suggests that, "the distinction thus established at the heart of a single concept between the intrinsic properties of an object [...] and its potential for action [on other objects] seems to resemble the Platonic distinction between the Same and the Other [...]"[9]. For Lautman, then, these mathematical items (equality, multiplication, unity, length and absolute value) all have, in some sense, one foot in each of two camps. We shall see this pat-

[8]But cf. Trevaskis's argument that there is more to the philosopher's technique than the method of division (Trevaskis, 1967).

[9]"La distinction qui s'ètablit ainsi au sein d'une même notion entre les propriétés intrinsèques d'un être ou d'une notion et ses possibilités d'action nous semble s'apparenter à la distinction platonicienne du Même et de l'Autre qui se retrouvent dans l'unité de l'être" (Lautman, 2006, p. 79). (Translation note: this translation is a little free in order to preserve Lautman's special sense of *notion*).

tern again in the fifth chapter of the *Essay on the notions of structure and existence in mathematics*, in which Lautman explores another collection of mathematical 'mixtures'. Notice, though, that the pairs of notions in this list are not pairs of conceptual opposites. He has this in common with the Eleatic philosopher; swimming is not the opposite of walking, nor is fish the opposite of fowl. The fact that these pairs are *not* conceptual opposites raises the question why the Eleatic philosopher divides categories into pairs (rather than triples, quadruples, *etc.*), with all the resulting awkwardness and arbitrariness. In another late work that Lautman mentions, *Timaeus*, Plato divides living creatures into four classes according to habitat: gods in heaven, birds in the air, land animals and water animals (39–40). Similarly in *Philebus*, when Socrates describes the method of division he requires only that a category be divided into a finite number of sub-categories (16D). The view that dialectics relates notions in pairs is indeed present in the *Sophist*, but only in the figure of the generic Eleatic philosopher. It does not seem to have been Plato's doctrine.

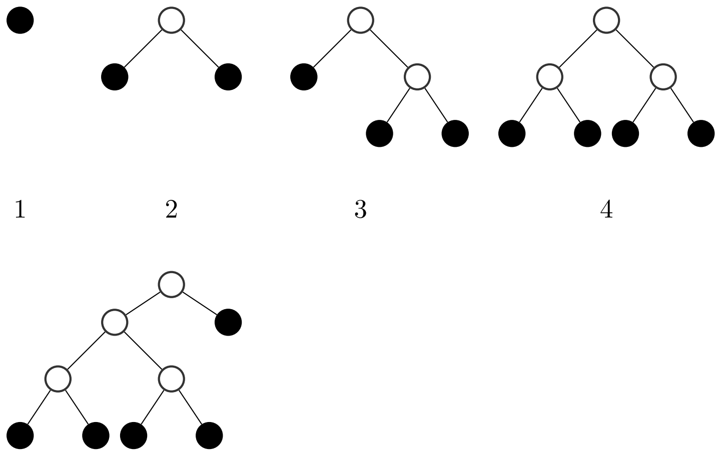

FIGURE 1. Lautman's Tree-diagrams.

The final reference to the *Sophist* is rather indirect. Lautman ends the *Essay on the notions of structure and existence in mathematics* with a gesture towards the thought that there is a developmental order among dialectical Ideas and from Ideas to mathematics. Lautman appeals to the works of Oscar Becker and Julius Stenzel on number in Plato and Aristotle. Lautman supplies a diagram taken from Stenzel (cf. Figure 1). Iterations of the Ideas 'one' and 'pair' produce 'Idea-numbers' (represented by the tree-diagrams), which in turn engender arithmetical numbers (represented by the black dots). Lautman's discussion is confused and inconclusive. He reproduces this diagram in the main text *and* in the footnotes, and mentions some reservations on Becker's part without discussing them (Lautman, 2006, p. 230, Figure 9). In any case, as Lautman acknowledges, Becker and Stenzel were both reading Plato through Aristotle (this was also Heidegger's procedure).[10] Having made this gesture, Lautman then turns to the relationship between mathematics and physics. The brief, inconclusive discussion with its pointlessly repeated diagram suggests some haste and dissatisfaction on Lautman's part.

1.2 *Philebus*

Lautman mentions *Philebus* once, in the quotation given above, as a source for the method of division. In this dialogue, Philebus, one of Socrates's young companions, holds that the good for man is pleasure. Socrates sets out to contest this, and to argue that intelligence (including knowledge and judgement) is better than pleasure. Before proceeding to his argument, Socrates makes a methodological digression. He describes the method of division (16C–17A), and insists that scientific understanding of a topic requires knowledge of the structure generated by successive distinctions. Unlike the Eleatic philosopher, Socrates allows that a category may divide into more than two subclasses; all he insists is that the number of subclasses should be finite (16D).

Early in the dialogue, Socrates points out that a life of pure intellectual activity is not suitable for men (21E). The good life for men must include some sensuous enjoyment as well as intellectual activity—though this being Plato, the intellectual side has priority. The crucial point is that the good for men is a mixture of heterogeneous elements (sensual and intellectual). This would present a paradox, if the method of division had the Aristotelian purpose of establishing a taxonomy. In the 'Aristotelian logic of genus and species', an object that seems to belong to two different species would be a counterexample to the taxonomy (as, for example, the duck-billed platypus threatens the category 'mammal'). In contrast, a Platonic system of Ideas is somehow prior to and independent of the objects that participate in those Ideas. An object can participate in more than one Idea (for example, a physical object might be both red and round). This mixing of Ideas occurs in other later dialogues. It is one of the principal explanatory motifs in the physics of *Timaeus* (34–35, 59–61), and we have already seen it in the *Sophist*.

For our purposes, the significant outcome of *Philebus* is that every human life must embody a mixture of sensual and intellectual goods. Pre-

[10]Heidegger (1925, p. 8). Lautman was not the only one to find Stenzel's reading of Plato on arithmetic more suggestive than clear. Cf. Cornford (1924); Shorey (1924).

cisely which goods and how they connect will vary from life to life. Perhaps someone's enjoyment of wine will develop an intellectual aspect as connoisseurship. To take a different example, intellectual work may offer some pleasures and satisfactions (though this is not, in Plato, the reason why it is good). One might imagine a life in which intelligence and pleasure were entirely separate departments, though it is hard to imagine desiring such an existence. Every human life will embody this dialectic in some way, and on philosophical examination will disclose it. Lives that lack one or other element must show that lack as an inadequacy or discontent. Indeed, we would need this dialectic of the sensual and the intellectual in order to understand the distempers and changes within a particular life. The background dialectical structure explains why a life given excessively to either sensuous pleasures or intellectual goods would be unsatisfactory.

Philebus, then, gives us an ethical analogy of Lautman's account of dialectical Ideas in mathematics. In Lautman's terminology, pleasure and intelligence are 'notions' and the possibility of relations between them is an 'Idea'. This dialectical structure does not specify which pleasures and thoughts will actually obtain. As Lautman says of mathematical Ideas, "As they are merely sketches of eventual positions, [Ideas] do not necessarily entail the existence of particular beings capable of sustaining the relations that the Ideas outline".[11] These notions come into relation through the interplay of particular thoughts and pleasures, and there is no predicting the detail of that interplay from the bare dialectical structure. The wine connoisseur's knowledge inflects his pleasure in boozing. Pleasure and intelligence relate quite differently (but no less intimately) in the rare but precious moments of insight in the work of a scientist. Similarly, Lautman maintains there is an indefinite variety of ways in which any dialectical relation between notions might manifest itself in actual mathematics, and it is not the business of philosophers to attempt to predict or circumscribe these relations (Lautman, 2006, p. 229).

1.3 *Timaeus*

Lautman's two references to *Timaeus* (Lautman, 2006, pp. 231 & 267) both remind us that for Plato, the creation of a material world is possible only if there is already a 'geometrically ordered receptacle' called 'place'.[12] Crucially, *different* objects may (at different times) occupy the *same* place. Thus, 'place' depends for its intelligibility on an anterior dialectical pair:

[11] "Étant seulement dessin de positions éventuelles, elles n'entraînent pas forcément l'existence d'êtres susceptibles de soutenir entre eux les relations qu'elles ébauchent" (Lautman, 2006, p. 243).

[12] "[...] le réceptacle d'une qualification géométrique" (Lautman, 2006, p. 231); translation note: the more literal "receptacle of a geometric qualification" makes little sense); "le lieu" (Lautman, 2006, p. 267). Cf. *Timaeus* §§48–9.

same/other. As we saw above, Lautman regards 'the distinction [...] between the intrinsic properties of an object [...] and its potential for action' on other objects as an expression of the same/other relation. In both cases, reference to Timaeus enables Lautman to shift from philosophy of mathematics to philosophy of physics. Lautman argues that the natural world is mathematically intelligible because the same dialectical structures underlie both physics and mathematics. He offers enantiomorphic crystals as an example of a physical phenomenon in which dialectical opposites (in this case, symmetry and dissymmetry) are 'mixed'. (This paper will not further discuss Lautman's philosophy of physics.) Here, as in his allusion to Stenzel's work on number, Lautman is trying to illustrate his thought that the intelligibility of mathematics and physics requires a prior dialectical order. In both cases, his exposition stumbles over Plato's inability to say what 'dialectical priority' means.

For Lautman, then, the method of division reveals dialectical 'notions' (in his special sense of the word), and with them the Ideas of relations between these notions. However, Lautman does not offer sequences of distinctions. His notions do not form tree-shaped accounts like those of the philosopher in the *Sophist*. As the quotation at the head of this section suggests, what he takes from these later dialogues is the thought that a particular can participate in heterogeneous categories simultaneously. In some of his examples, the notions 'mixed' in a mathematical theory are merely different (such as ordinal and closure), while in others they are opposites (as in the cases where he sees mixtures of finite and infinite mathematics).

2 Plato does not suffice

Lautman scattered references to Plato throughout his works; Heidegger, on the other hand, does not feature anywhere in his writing other than the discussion in *New research on the dialectical structure of mathematics* and implicitly in some brief remarks in the conclusion to *Essay on the notions of structure and existence* (Lautman, 2006, pp. 228–229).[13] We may therefore suppose that Lautman turned to Heidegger in order to solve a particular problem in his overall Platonism.[14] Moreover, the Heideggerian text that he refers to, *On the Essence of Ground*, is a meditation on the 'ontological difference' between the 'ontic' concepts employed in the sciences and the

[13]The sole exception is in a short piece of 1933 *Considérations sur la logique mathématique*. But here he discusses the use that the intuitionists made of phenomenology and makes no commitment of his own: "Les intuitionnistes se rattachent par là aux phénoménologues disciples de Husserl, Heidegger, et Oscar Becker" (Lautman, 2006, p. 43).

[14]Which is not to suggest that Lautman chose Heidegger arbitrarily, given his references to Plato; Heidegger prefaced *Being and Time* with a quotation from the *Sophist* (244a), and he devoted his lectures of 1924/25 to that same dialogue.

underlying ‘ontological’ concepts disclosed by phenomenology. The relation between dialectics and mathematics was clearly problematic for Lautman. On one hand, he was committed to his Platonist view that Ideas are somehow prior to the matter that they dominate, and which participates in them. In a talk given in 1937, Lautman claims that, “The reality inherent in mathematical theories is due to their participation in an ideal reality which dominates mathematics, but which cannot be known except through mathematics”[15]. He knew that the logical empiricist mainstream would regard his view as a mystification, “as obscure as the mystical beliefs of primitives in the participation of subjects in objects of which Mr. Lévy-Bruhl speaks”.[16] He retorts that, on the contrary, empiricism (whether Aristotelian or Viennese) separates thought from experience and thus makes a mystery of the fact that we find nature mathematically intelligible. Moreover, a tautological view of mathematics separates the discovery of truth from the quest for reality (since tautologies do not require reference to any reality). Empiricism, he thought, deprives science of its spiritual dignity and value. Thus, it is scientifically and spiritually vital to insist on the reality of dialectical notions and the Ideas of their possible relations prior to their realisation in particular cases. On the other hand, notions only come into relations with each other when ‘mixed’ in particulars. Towards the end of the *Essay on the notions of Structure and Existence*, Lautman characterises Ideas of possible relations between notions as ‘problems’ or ‘questions’ and actual (realised) relations between notions as ‘logical schemas’:

> The logical schemas that we have described are not prior to their realisation at the heart of a theory; what is lacking from [...] the extra-mathematical intuition of the urgency of a logical problem is that it must have material to dominate, for the idea of possible relations to give birth to a scheme of real relations.[17]

Before the development of the mathematical theory that solves the problem, there is only “the experience of the urgency of problems”.[18] However, this formulation makes it sound as if we are concerned with the psychology

[15]“La réalité inhérente aux théories mathématiques leur vient de ce qu’elles participent à une réalité idéale qui est dominatrice par rapport à la mathématique, mais qui n’est connaissable qu’à travers elle” (Lautman, 2006, pp. 67–68).

[16]“[...] aussi obscures que les croyances mystiques à la participation du sujet à l’objet chez les primitifs dont parle M. Lévy-Bruhl” (Lautman, 2006, p. 64).

[17]“Les schémas logiques que nous avons décrits ne sont pas antérieurs à leur réalisation au sein d’une théorie; il manque en effet à ce que nous appelons plus haut l’intuition extra-mathématique de l’urgence d’un probléme logique, une matière à dominer pour que l’idée de relations possibles donne naissance au schéma de relations véritables” (Lautman, 2006, p. 229).

[18]“Le seul élément *a priori* que nous concevions est donné dans l’expérience de cette urgence des problèmes[...]” (Lautman, 2006, p. 229).

of mathematicians. That is not what Lautman had in mind. In the introduction to the *Essay on structure*, he concludes a discussion of Hilbert and Brunschvicg by insisting that, "Between logical deduction and the psychology of the mathematician, there must be space for an intrinsic characterisation of reality".[19]

What he requires, then, is a philosophical idiom in which this feeling for the urgency of a problem is more than a *mere* feeling. For, if this sense of urgency is no more than a psychological urge, then its content cannot have the logical significance that Lautman's Platonism requires. At the same time, this 'extra-mathematical intuition of the urgency of a logical problem' cannot be a mysterious sensitivity to a world of Ideas that exist prior to the activity of mathematicians. We have already seen that he rejects that kind of naïve Platonism. It is to fill this need that Lautman turns to Heidegger.

3 Heidegger

Lautman was familiar with *Being and Time*,[20] but he appeals to a much shorter work of Heidegger's: *On the Essence of Ground* (1928).[21] In this lecture, Heidegger tries to clarify his distinction between 'ontological' and 'ontic' (or equivalently, between 'being' and 'beings'). He does this through a meditation on the history of philosophy that resists summary, but the central thought is as follows. From Kant, we learn that metaphysics (or in Heidegger's terminology, 'ontology') is not directly concerned with what the world is like 'in itself'. Rather, ontology primarily reveals the deep structure of how we go at the world and go on in it. In this exposition, 'the world' should be read in something like the sense it has when one says that people who routinely practice sympathetic magic 'live in a different world' from people who routinely pursue their ends by modern scientific means. In this sense, only humans live in 'the world'. Cats and dogs occupy physical space, but they do not have a deep and largely inarticulate sense of what the world is like and how it works that shapes and guides their activities. The crucial feature of humans is that we enquire. Our questions may be practical (such as "why is my knee throbbing?" or "will I have to mix some more cement to finish this wall?") or they may be part of an advanced science. Scientific or not, every question has built in some assumptions about the form of the answer. A sleeper woken by a noise might ask, "what's that?" or "who's there?" or "did I imagine that?", depending on her expectations and cast of mind. This is true even of questions that seem to make no assumptions, such as "why is there something rather than nothing?" We did not have to ask

[19]"Entre la psychologie du mathématicien et la déduction logique, il doit y avoir place pour une caractérisation intrinsèque du réel" (Lautman, 2006, p. 129).

[20]Cf. his reference to it, (Lautman, 2006, p. 240).

[21]I am grateful to Nicholas Joll for his unstinting help with my reading of Heidegger.

about *things*. We might have asked, "Why is there stuff rather than void?" Scientific disciplines have their characteristic ways of going at the world: the modern physicist asks questions in the language of mathematics. This would be unintelligible to earlier students of nature who 'lived in another world' in the phenomenological sense (even if they understood the mathematics). Thus, our questions always reveal something of the deep structural features of the world as we take it.

What then of the 'ontological difference'? In a passage that Lautman quotes, Heidegger declares:

> The prior determination of the being (what-being and how-being) of nature in general is anchored in the "fundamental concepts" [*Grundbegriffe*] of the relevant science. In such concepts, space, place, time, motion, mass, force, and velocity are delimited, for example, and yet the essence of time or motion does not become an explicit problem. [...] The fundamental concepts of contemporary science neither contain the "proper" ontological concepts of the being of those beings concerned, nor can such concepts be attained merely through a "suitable" extension of these fundamental concepts.[22]

In other words, even the most fundamental concepts that scientists use are merely ontic. The corresponding ontological concepts lie outside the conceptual resources of science. Lautman insisted that the same is true of the dialectical notions and ideas that he discerns at work within mathematics. "Dialectic", he says, "is not part of mathematics, and its notions have no connection with the primitive notions of a theory".[23] Rather, dialectic is ontologically prior to mathematics in Heidegger's sense of 'ontological'. In the conclusion of *Essay on the notions of structure and existence in mathematics*, Lautman describes his philosophy of mathematics as phenomenological enquiry into the extra-mathematical intuition of or concern (*souci*) with the 'urgency' of a logical problem. We should read *souci* here as *Sorge* (concern or care) in the Heideggerian sense.

Heidegger's phenomenology is primarily concerned with our habits of mind and expectations as revealed in our questions (or rather, in our practices of enquiry). However, this enquiry into the structure of our active, questioning subjectivity also reveals the order of objective reality. Heidegger takes from Kant the thought that the coherent order of human subjectivity and the coherent order of the world are two sides of the same fact. Somehow, at an inexpressibly deep level, three aspects come together: the structure of our subjectivity, our busy activity using things to work on other things and the deep structures that we find embodied in the world.

[22] Heidegger (1967, p. 104–105). Quoted by Lautman (2006, p. 241).

[23] "La dialectique ne fait pas partie des mathématiques, et ses notions sont sans rapport avec les notions primitives d'une théorie" (Lautman, 2006, p. 242).

Lautman's Heideggerian account of the objectivity of mathematics seems to come to this: mathematical theories 'participate' in dialectical Ideas, in the sense that they relate dialectical notions. The Ideas pose vague and nebulous questions (in that they suggest the possibility of notional relations), to which mathematics supplies precise and detailed answers. In this sense, the Ideas call the mathematical theories into existence (though we should not expect to discern the Ideas doing the calling until after the mathematical theory is complete.) Thus, the mathematical theories depend for their objectivity on the Ideas. The Ideas are objective in the sense that they are part of the deep structure of our engagement with the world, which means that they are part of the deep structure of us, or, what comes to the same thing, they are part of the deep structure of *our* world. Since our world is the only one we know, we may as well say that they are part of the deep structure of *the* world, so long as we remember that this insight is ontological, not ontic. It belongs to phenomenology, not to science.

4 Lautman's mathematical examples

Having sketched Lautman's view abstractly, I shall now consider his mathematical examples. The first of Lautman's two theses (*On the unity of the mathematical sciences*) takes as its starting point a distinction that Hermann Weyl made in his 1928 work on group theory and quantum mechanics. Weyl distinguished between 'classical' mathematics, which found its highest flowering in the theory of functions of complex variables, and the 'new' mathematics represented by (for example) the theory of groups and abstract algebras, set theory and topology (Lautman, 2006, p. 83–84). For Lautman, the 'classical' mathematics of Weyl's distinction is essentially analysis, that is, the mathematics that depends on some variable tending towards zero: convergent series, limits, continuity, differentiation and integration. It is the mathematics of arbitrarily small neighbourhoods, and it reached maturity in the nineteenth century. On the other hand, the 'new' mathematics of Weyl's distinction is 'global'; it studies the structures of 'wholes' (Lautman, 2006, p. 84). Algebraic topology, for example, considers the properties of an entire surface (how many holes?) rather than aggregations of neighbourhoods. Having quoted and illustrated Weyl's distinction, Lautman re-draws it:

> In contrast to the analysis of the continuous and the infinite, algebraic structures clearly have a finite and discontinuous aspect. Though the elements of a group, field or algebra (in the restricted sense of the word) may be infinite, the methods of modern algebra usually consist in dividing these elements into equivalence classes, the number of which is, in most applications, finite.[24]

[24]"[...] en opposition à l'analyse du continu et de l'infini, les structures algébriques

The chief part of Lautman's 'unity' thesis is taken up with four examples[25] in which theories of modern analysis (that is to say, analysis as practiced in the twentieth century) depend in their most intimate details on results and techniques drawn from the 'new', algebraic side of Weyl's distinction. In these four cases, algebra comes to the aid of analysis. Thus, Lautman transforms a broad historical distinction (between the local, analytic, continuous and infinitisic mathematics of the nineteenth century, and the new, 'global', synthetic, discrete and finitistic style) into a family of dialectical dyads (local/global, analytic/synthetic, continuous/discrete, infinitistic/finitistic). These pairs are not empty oppositions. They find their content in the details of mathematical theories that, though they belong to analysis, sometimes employ a characteristically algebraic point of view. In other words, the methods are algebraic but the results belong to analysis.[26] By this point, we have left the nineteenth century behind, and are concerned with analytic/algebraic 'mixtures' in contemporary (twentieth century) mathematics.

In his other major thesis, *Essay on the notions of structure and existence in mathematics*, Lautman gives his dialectical thought a more philosophical and polemical expression. Six chapters compose the body of this second thesis, the first three on 'structural schemas' (*schémas de structure*), the second three on 'origination schemas' (*schémas de genèse*). The three structural schemas are: local/global, intrinsic properties/induced properties and the (unfortunately titled) 'ascent to the absolute'.[27] The first two of these three schemas are pairs of the sort we saw in Lautman's 'unity' thesis. The 'ascent to the absolute' is a different sort of pattern; it involves a progress from mathematical objects that are in some sense 'imperfect', towards an object that is 'perfect' or 'absolute'. His two mathematical examples of this 'ascent' are: class field theory, which 'ascends' towards the absolute class field, and the covering surfaces of a given surface, which 'ascend' towards a simply-connected universal covering surface. In each case, there is a corre-

ont un aspect nettement fini et discontinu. Quelle que soit l'infinité des éléments qui constituent un groupe, un corps, une algèbre (au sens restreint du mot), les méthodes de l'algèbre moderne consistent le plus souvent à imposer à ces éléments une division en classes d'éléments équivalents, et à substituer ainsi à un ensemble infini la considération d'un nombre de classes qui, dans les applications, est le plus souvent fini" (Lautman, 2006, pp. 86–87).

[25]Dimensional decomposition in function theory; non-Euclidian metrics in analytic function theory; non commutative algebras in the equivalence of differential equations; and the use of finite, discontinuous algebraic structures to determine the existence of functions of a continuous variable (Lautman, 2006, p. 87).

[26]"[...] il est possible de retrouver dans les théories modernes de l'analyse les points de vue qui caractérisent l'algèbre[...] théories dont les méthodes sont algébriques mais les résultats s'étendent à l'analyse" (Lautman, 2006, p. 121).

[27]*La montée vers l'absolu*. The Hegelian resonance of 'the absolute' is a red herring; it seems to have deceived Bernays (1940, p. 20).

sponding sequence of nested subgroups (with the trivial subgroup mapping to the 'absolute' class field or surface), which induces a 'stepladder' structure on the 'ascent'. (Lautman introduces this idea with a brief discussion of the Galois correspondence, (Lautman, 2006, pp. 166–168).) This dialectical pattern is rather different to the others. The earlier examples were of pairs of notions (finite/infinite, local/global, *etc.*) and neither member of any pair was inferior to the other.[28] As we saw, Lautman argues that on some occasions, finite mathematics offers insight into infinite mathematics (think for example of the use of finite integer fields in the study of infinitely numerous natural numbers). In mathematics, the finite is not a somehow imperfect version of the infinite. Similarly, the 'local' mathematics of analysis may depend for its foundations on 'global' topology (as Lautman argues, Lautman, 2006, p. 85), but the former is not a botched or somehow inadequate version of the latter. Lautman introduces the section on the 'ascent to the absolute' by rehearsing Descartes's argument that his own imperfections lead him to recognise the existence of a perfect being (God). Man (for Descartes) is not the dialectical opposite of or alternative to God; rather, man is an imperfect image of his creator. In a similar movement of thought, according to Lautman, reflection on 'imperfect' class fields and covering surfaces leads mathematicians up to 'perfect', 'absolute' class fields and covering surfaces respectively. In short, the 'ascent to the absolute' introduces a different dialectical structure from the pairs of notions we saw hitherto. It has nothing in common with the patterns found in the three Platonic dialogues.

The three origination schemas are titled 'Essence and existence', 'Mixtures' and 'On the exceptional character of existence'. In the first two of these chapters, the structure of a mathematical domain gives rise to new mathematical objects; in the third chapter, Lautman considers cases where an object is shown to exist in virtue of exceptional properties that distinguish it from an established set of objects. Lautman does not address directly the general question of the metaphysical status of mathematical objects. He is, rather, interested in the way that mathematical structures and objects grow out of already existing mathematics. As he explains, the roles of 'originating structure' and 'created object' are relative; objects that owe their existence to the structure of another domain may themselves come to constitute the originating structure for some further class of entities (Lautman, 2006, p. 187). Part of his purpose is to oppose the view that there is nothing more to mathematical existence than the consistency of an axiom system. Lautman spends some time rehearsing the familiar technical difficulties entailed in attempts to prove the consistency of a system (Lautman, 2006, pp. 179–187). But his real claim is that mathematical entities do

[28] But cf. also (Barot, 2008, p. 12). Recall too that for Plato, the intellectual is superior to the sensual.

not depend for their existence on apparently arbitrary decisions to explore some sets of axioms but not others. Rather, mathematicians create new mathematical structures in the course of answering questions latent in the underlying extra-mathematical dialectical order. Here too, the dialectical patterns that he discerns are more richly varied than his references to Plato would suggest.

5 Blurring the ontological difference

Lautman concludes the *Essay on the notions of structure and existence in mathematics* with some remarks that, though they do not mention Heidegger by name, are clearly of a piece with the explicit discussion of Heidegger in *New research on the dialectical structure of mathematics* (Lautman, 2006, pp. 228–229). After a brief discussion of Plato (Lautman, 2006, pp. 230–234), Lautman ends his thesis with a statement of his credo, which he held to be true of mathematics and physics alike:

> The nature of reality, its structure and the conditions of its origination cannot be known except by returning to the Ideas that science embodies in its inner relations.[29]

As we saw, Lautman appeals to Heidegger in order to explain the relation between dialectics and mathematics. The whole point of *On the Essence of Ground* is to insist on the ontological difference, that is, on the distinction between the ontological and the ontic. The division of labour between the scientist and the philosopher depends on this distinction. The scientist uses ontic concepts to establish ontic truths; the philosopher reveals the corresponding ontology. Lautman insists on the distinction between dialectics and mathematics. If dialectic tries to find its own solutions to the problems it expresses, it will "mimic mathematics with such a collection of subtle distinctions and logical tricks that it will be mistaken for mathematics itself".[30] This, he suggests, is the fate of the logicism of Frege and Russell. Dialectical notions and ideas must find expression in mathematical examples. Expression in mathematical examples subjects an Idea to "a whole train of specifications, limitations and exceptions with which mathematical theories are constructed and confirmed".[31] So, for example, we might

[29] "La nature du réel, sa structure et les conditions de sa genèse ne sont connaissables qu'en remontant aux Idées dont la science incarne les liaisons" (Lautman, 2006, p. 234).

[30] "Une dialectique qui s'engagerait dans la détermination des solutions que ces problèmes logiques peuvent comporter, se verrait entraînée à constituer tout un ensemble de distinctions subtiles et d'artifices de raisonnement qui imiteraient a ce point les mathématiques, qu'elle se confondrait avec les mathématiques elles-mêmes" (Lautman, 2006, p. 228).

[31] "Il faut ensuite, pour que l'exemple supporte l'Idée, apporter à celle-ci tout un cortège de précisions, de limitations et d'exceptions où s'affirment et se construisent les théories mathématiques" (Lautman, 2006, p. 243).

look at the various mathematical concepts of completeness and closure, and recognise in them mathematical versions of the vague (and hence presumably dialectical) notion that a complex item might be self-sufficient or *sui generis*. To recall one of his detailed examples, Lautman invites us to see the mathematical relations between the intrinsic and relational properties of mathematical objects as a mathematical specification of the dialectic of same and other.

But now we have a problem. How are we to distinguish between the legitimate activity of seeking mathematical answers to dialectical questions, and the mistaken activity of making dialectics imitate mathematics? After all, historically, mathematics does not have fixed borders. For example, Euler thought that the Königsberg bridges problem lay outside mathematics, because "the solution is based on reason alone, and its discovery does not depend on any mathematical principle".[32] Formal logic lay outside mathematics for over two millennia (if we measure from Aristotle's *Prior Analytics* to Boole's 1847 *Mathematical Analysis of Logic*). Aside from the authority of Heidegger, Lautman's distinction between dialectics and mathematics depends on the "essential insufficiency"[33] of dialectical Ideas, that is, the fact that they cannot be understood except through the development of mathematical theory. However, this is also true of undeveloped or primitive mathematical concepts. The primitive concepts of continuity and infinity posed questions that were only properly answered through the contemplation of mathematical examples and the articulation of mathematical theories. For all that, continuity and infinity are clearly mathematical concepts, however primitive.

Lautman's own examples suggest that the line between dialectics and mathematics is neither clear nor stable. Look again at the diagram he takes from Stenzel (Figure 1). Lattices were not mathematical objects in Plato's day, but they are now. Are we to suppose that the underlying dialectical structure of Plato's arithmetic is itself an example of a mathematical concept (namely, lattice), which presumably has a dialectical basis of its own? Lautman explicitly rejects such regresses (Lautman, 2006, p. 232).

The second part of *New research on the dialectical structure of mathematics* is a pair of case studies that pick up the contrast between analysis and algebra that we first met in *Essay on the unity of the mathematical sciences*. In that early essay, we saw algebra coming to the aid of analysis. In these two cases, we see analysis (the mathematics of continuity) supplying proofs to number theory. The second case supports Lautman's argument for the unity of mathematics rather well: it is the use of the Rie-

[32]Letter 590 in Euler's *Opera Omnia*: quoted from Wilson (2008, p. 15). Euler wondered whether it might be what Leibniz meant by 'geometry of position'.

[33]"Insuffisance essentielle" (Lautman, 2006, p. 243).

mann zeta-function to investigate the density of primes. The first case is rather artificial: it is Hecke's proof of quadratic reciprocity. It is artificial because (as Lautman acknowledges) there is no need to call on analysis to prove this theorem. Of Hecke's proof of quadratic reciprocity, Lautman writes:

> The analytic tool, that is to say, functions, serves to demonstrate an arithmetical result because the structure of the tool and that of the result both participate in the same dialectical structure, which poses the problem of the reciprocity of roles between mutually inverse elements.[34]

This presents two problems for the distinction between dialectics and mathematics. First, reciprocity is a kind of symmetry. The symmetry that obtains between these mutually inverse elements hardly requires the apparatus of group theory, but it is, nevertheless, a mathematical concept, as indeed are the relata. Second, the proper relationship between dialectics and mathematics appears to have been reversed. In a letter to the mathematician Maurice Fréchet, Lautman explained:

> It is insofar as a mathematical theory supplies an answer to a dialectical problem that is definable but not resolvable independently of mathematics that the theory seems to me to participate, in the Platonic sense, in the Idea with regard to which it stands as an Answer to a Question.[35]

In principle, then, dialectics stands to mathematics as question to answer, but here a mathematical question (why does this body of analytic theory serve to prove that arithmetical result?) gets a dialectical answer. In any case, 'participates in the same dialectical structure as' is a symmetric relation, but the tool-result relation is not. He gives other examples in *New research on the dialectical structure of mathematics* in which (he claims) "the convergence of different mathematical theories results from the affinity of their dialectical structures",[36] but he elsewhere gives examples of mathematical theories that share dialectical structures (such as all the same/other examples) but do not show any sign of convergence.

[34]"l'outil analytique, c'est-à-dire les fonctions, sert à démontrer un résultat arithmétique, parce que la structure de l'outil et celle du résultat participent l'une et l'autre d'une même structure dialectique, celle que pose le problème de la réciprocité de rôles entre éléments inverses l'un de l'autre" (Lautman, 2006, p. 248).

[35]"C'est dans la mesure où une théorie mathématique apporte une réponse à un problème dialectique définissable mais non résoluble indépendamment des mathématiques que la théorie me paraît participer, au sens de Platon, à l'Idée vis-à-vis de laquelle elle est dans la même situation que la Réponse par rapport à la Question" (Lautman, 2006, p. 260).

[36]"la convergence des théories mathématiques différentes résulte de leur affinité de structure dialectique" (Lautman, 2006, p. 250).

In short, the claim that science and phenomenology treat of different concepts collapses in mathematical practice.[37] Symmetry (for example) is a mathematical concept, but it can also function as a dialectical notion in Lautman's sense—it is one of the notions that shape our questions. If it does so explicitly, it may also function as a heuristic in the sense of Polya. Lautman does not mention heuristics; rather, he insists that we should not expect to discern a dialectical question in advance of arriving at its mathematical answer. However, this overlooks the fact (which Lautman elsewhere insists on) that the same dialectical notions and Ideas may feature in different mathematical theories. As a notion or Idea recurs in various mathematical theories, it may become an explicit part of the mathematical culture and thus begin to function heuristically. Recognising that a concept can serve on either side of the dialectical/mathematical distinction would be consistent with what we find in the later Plato. The 'same/other' dyad may have a deep ontological role, lending intelligibility to concepts as diverse as 'place' and 'intrinsic/extrinsic', but it also has a function in unremarkable empirical questions like "Is that the same dog as I saw yesterday?". Consequently, if we wish Lautman to enrich our own philosophy, the first move should be to give up the 'ontological distinction'. This thought can even find some support in Heidegger. In a 1936 lecture *Modern Science, Metaphysics and Mathematics*, he writes:

> The greatness and superiority of natural science during the sixteenth and seventeenth centuries rests in the fact that all the scientists were philosophers. They understood that there are no mere facts, but that a fact is only what it is in the light of the fundamental conception [...] the present leaders of atomic physics, Niels Bohr and Heisenberg, think in a thoroughly philosophical way [...](Heidegger, 1962, p. 272).

It is hard to see what advantage philosophy could bring to science if the ontological distinction stands between them. Heidegger's claim that great scientists are also philosophers suggests that they do not respect the 'ontological distinction' in their practice.

This modification would also give Lautman a reply to a criticism from one of his closest colleagues. At a meeting in February 1939, Lautman insisted that the objectivity of mathematical theories depends on their participation in non-mathematical Ideas that dominate them. Also present was Jean Cavaillès, who remarked, "Personally, I recoil from positing something else which would dominate the actual thought of mathematicians, I see necessity in the problems [...]"[38]. Giving up the 'ontological distinction' would

[37] For an independent argument with a similar conclusion, cf. Barot (2008, p. 14–17)

[38] "Personnellement je répugne à poser une autre chose qui dominerait la pensée effective du mathématicien, je vois l'exigence dans les problèmes[...]" (Lautman, 2006, p. 263).

allow Lautman to reply that dialectical Ideas do indeed dominate mathematical theories, but the Ideas, the theories and the domination are all part of mathematical thinking. Thus, talk of 'domination' notwithstanding, no extraneous constraint cramps the thought of mathematicians.

Bibliography

Barot, E. (2008). L'Objectivité Mathématique selon A. Lautman: entre Idées dialectiques et réalité physique. *Cahiers François Viète*, 6:3–28.

Bernays, P. (1940). Review of *Essai sur les notions de structure et d'existence en mathématiques* and *Essai sur l'unité des sciences mathématiques dans leur développement actuel* by Albert Lautman. *Journal of Symbolic Logic*, 5(1):20–22.

Cornford, F. M. (1924). Review of *Zahl und Gestalt bei Platon und Aristoteles* by Julius Stenzel. *Classical Review*, 38(7–8):209.

Heidegger, M. (1925). Platon: Sophistes. Lecture course Wintersemester 1924/25; published as Schüßler (1992). Page numbers refer to the English translation (Heidegger, 2003).

Heidegger, M. (1962). *Die Frage nach dem Ding. Zu Kants Lehre von den transzendentalen Grundsätzen.* Max Niemeyer, Tübingen. Page numbers refer to the English excerpt in Heidegger (1993).

Heidegger, M. (1967). *Wegmarken.* Vittorio Klostermann, Frankfurt a. M. Page numbers refer to the English translation (Heidegger, 1998).

Heidegger, M. (1993). *Basic writings.* Routledge, London.

Heidegger, M. (1998). *Pathmarks.* Cambridge University Press, Cambridge.

Heidegger, M. (2003). *Plato's Sophist.* Indiana University Press, Bloomington, IN.

Lautman, A. (2006). *Les mathématiques, les idées et le réel physique.* Vrin, Paris. Introduction and biography by Jacques Lautman; introductory essay by Fernando Zalamea; preface to the 1977 edition by Jean Dieudonné.

Schüßler, I., editor (1992). *Martin Heidegger. Gesamtausgabe. II. Abteilung. Vorlesungen 1919–1944. Band 19. Platon: Sophistes.* Vittorio Klostermann, Frankfurt a. M.

Shorey, P. (1924). Review of *Zahl und Gestalt bei Platon und Aristoteles* by Julius Stenzel. *Classical Philology*, 19(4):381–383.

Trevaskis, J. R. (1967). Division and its relation to dialectic and ontology in Plato. *Phronesis*, 12(2):118–129.

Wilson, R. (2008). Euler's combinatorial mathematics. *Bulletin of the British Society for the History of Mathematics*, 23(1):13–23.