EXERCISE SHEET 8 - SOLUTIONS

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Definition 0.1. Let $C$ be a cocomplete category. A set of objects $S \subseteq \text{obj}(C)$ is called a generating set of $C$, if a map $f : x \to y$ in $C$ is an isomorphism if and only if the induced map

$$f_* : \text{Hom}_C(s, c) \to \text{Hom}_C(s, d)$$

is an isomorphism for all $s \in S$.

Exercise 1. Let $C$ be a cocomplete category with generating set $S$ consisting of compact objects. Show that the compact objects of $C$ span the smallest full subcategory of $C$ that contains $S$ and is closed under finite colimits.

Deduce the following:

1. A set is compact in $\text{Set}$ if and only if it is finite.
2. A simplicial set is compact in $\text{sSet}$ if and only if it is finite.
3. Let $R$ be a ring. An $R$-module is compact in $\text{Mod}(R)$ if and only if it is finitely presented.

Can you also identify the tiny objects in these categories?

Note that the third item in particular implies that the forgetful functor $\text{Ab} \to \text{Set}$ preserves filtered colimits (and since it has a left adjoint, it preserves all limits). It does of course not preserve coproducts!

Let me also warn you that the analogous quasi-categorical statement is not quite correct because the nerve of a finite category is not generally a finite simplicial set, which changes the meaning of finite colimit when passing from categories to quasi-categories! The buzzword is idempotent completion.

Solution. We first observe that finite colimits of compact objects are again compact, because finite limits and filtered colimits of sets commute, so

$$\text{Hom}_C(\text{colim}_{i \in I} F(i), \text{colim}_{j \in J} G(j)) = \lim_{i \in I} \lim_{j \in J} \text{Hom}_C(F(i), G(j)) = \lim_{i \in I} \text{colim}_{j \in J} \text{Hom}_C(F(i), G(j)) = \text{colim}_{j \in J} \lim_{i \in I} \text{Hom}_C(F(i), G(j))$$

whenever $I$ is finite, $F(i)$ is compact for all $i \in I$ and $J$ is filtered.

Now by $\mathcal{S}$ the smallest full subcategory of $C$ containing $S$ and closed under finite colimits.

We will show that $\mathcal{S}$ consists precisely of the compact objects in $C$.

We need to describe $\mathcal{S}$: Setting $S_0 = S$ and $S_{n+1}$ to consist of all objects that can be written as a finite colimit of objects in $S_n$ we have $\mathcal{S} = \bigcup_{n \in \mathbb{N}} S_n$: Since any finite diagram $I \to \bigcup_{n \in \mathbb{N}} S_n$ takes values in some $S_n$, so has a colimit in $S_{n+1}$ the category $\bigcup_{n \in \mathbb{N}} S_n$ is closed under finite colimits, and by construction $S_n \subseteq \mathcal{S}$ for all $n$.

One can easily trick oneself into thinking $\mathcal{S} = S_1$, but I don’t think this is the case in general (I don’t know a counterexample though; not every abelian group is the colimit of a diagram whose only value is $\mathbb{Z}$, but every finitely presented one is).
Now, by the first preliminary observation $S_n$ consists of compact objects for each $n$, so $\overline{S}$ does too. To see the converse we first show that any object of $\mathcal{C}$ is a filtered colimit of objects in $\overline{S}$.

From here the proof can be finished as follows. Take a compact object $X$ and write it as $\text{colim}_{j \in J} G(j)$ with $J$ filtered, and $G(j) \in \overline{S}$ for all $j \in J$. Then
\[
\text{Hom}_\mathcal{C}(X, X) = \text{Hom}_\mathcal{C}(X, \text{colim}_{j \in J} G(j)) = \text{colim}_{j \in J} \text{Hom}_\mathcal{C}(X, G(j)).
\]

But this means that the identity of $X$ factors through some $G(j)$, presenting $X$ as a retract of an object in $\overline{S}$. But any category that admits finite colimits is closed under retracts: If
\[
X \xrightarrow{i} Y \xrightarrow{p} X
\]
exhibits $X$ as a retract of $Y$, i.e. $pi = \text{id}_X$, then $X$ is readily checked to be the colimit (and in fact also limit!) of the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\text{id}_Y} & Y \\
\downarrow{i} & & \downarrow{p} \\
& & \\
\end{array}
\]
a so-called coequaliser: A map from $X$ to some other object $Z$ determines a map $Y \to Z$ by precomposition with $p$, and conversely a map $Y \to Z$ determines a map $X \to Z$ by precomposition with $i$. The composition of these processes takes any map $X \to Z$ to itself, and the same goes precisely for those maps $g: Y \to Z$ which coequalise $ip$ and $\text{id}_Y$, i.e. which satisfy $gip = g$. It is this statement which is false in quasi-categories, see below. In the case hand, however, we obtain $X \in \overline{S}$ as claimed.

To see that every object $X$ of $\mathcal{C}$ can be written as a filtered colimit of objects in $\overline{S}$ consider the essentially small category $\overline{S}/X$. The forgetful functor provides a diagram
\[
\overline{S}/X \longrightarrow \overline{S} \longrightarrow \mathcal{C}
\]
together with a natural transformation to the constant functor with value $X$. This yields a map
\[
\text{colim}_{Y \in \overline{S}/X} Z \longrightarrow X
\]
that I claim is an isomorphism and furthermore $\overline{S}/X$ is filtered. The latter statement is why we had to close $\overline{S}$ under finite colimits: Given a finite diagram $I \to \overline{S}/X$ the colimit of the composite $F: I \to \overline{S}/X \to \overline{S}$ still admits a map to $X$ and since $\overline{S}$ is closed under finite colimits this gives a cone of $F$ in $\overline{S}/X$; that is what it means to be filtered (and this argument shows, that any category with finite colimits is filtered). To see that the map to $X$ is an isomorphism, we have to use the assumption that $\overline{S}$ is a generating set: Because of it, it suffices to check that the map
\[
\text{Hom}_\mathcal{C}(T, \text{colim}_{Y \in \overline{S}/X} Y) \longrightarrow \text{Hom}_\mathcal{C}(T, X)
\]
is an isomorphism for every $T \in \overline{S}$. But since the index category is filtered and $T$ compact, this is the same as showing that
\[
\text{colim}_{Y \in \overline{S}/X} \text{Hom}_\mathcal{C}(T, Y) \longrightarrow \text{Hom}_\mathcal{C}(T, X)
\]
is an isomorphism. But that is obvious by direct inspection: It is surjective, since any map $f: T \to X$ on the right occurs as an index on the left and $\text{id}_T \in \text{Hom}_\mathcal{C}(T, T)$ is then send to $f$. If on the other hand two maps $f: T \to Y$ and $f': T \to Y'$ on the left have equal composite $T \to X$ on the right, then they define a map $Y \cup_T Y' \to X$, which we can regard as an index on the left (since $\overline{S}$ is closed under pushouts). The composition $T \to Y \cup_T Y'$ by construction defines the same point in the left hand colimit as either $f$ or $f'$, so we obtain injectivity.
We can now easily handle the examples, since we know compact generators in all cases: For the category of sets we can use $S = \{ \ast \}$, since $\text{Hom}_C(\ast, -)$ is the identity functor. In this case $S = S_1$ consists exactly of the finite sets.

For the case of simplicial sets, we use $S = \{ \Delta^n \mid n \in \mathbb{N} \}$: Generally, for the category $\text{Fun}(C^{\text{op}}, \text{Set})$ with $C$ small the representable functors form a generating set by Yoneda’s lemma (and the fact the pointwise invertible natural transformations are invertible!). In the case of simplicial sets we showed that the standard presentation of a functor $C^{\text{op}} \to \text{Set}$ as a colimit of representable can be simplified by removing the degenerate simplices, i.e. for any simplicial set $X$ we have

$$\text{colim} \Delta^n = X,$$

where $S(X)$ denotes full subcategory of $Y_{\Delta/X}$ spanned by the non-degenerate simplices. This shows that $S_1$ contains all finite simplicial sets. And since the category of these is clearly closed under finite colimits, we again have $S_1 = \overline{S}$ as claimed.

For case of $R$–$\text{Mod}$ we can use $S = \{ R \}$. Then since a finite presentation is a particular way of writing a module as a two-step colimit of $R$’s, $S_2$ contains all such. But again, finitely presented $R$-modules are evidently closed under finite colimits (concatenate presentations), so $S_2 = \overline{S}$. I’m not quite sure, whether generally $S_1 = \overline{S}$ in this case.

The tiny objects are given by $\ast$ in the case of sets and by the simplices in the case of simplicial sets. That these are tiny is immediate by inspection. That there are no other tiny objects follows by running the proof above with a set $S$ of tiny generators. In that case there is no need to pass to $\overline{S}$ and the arguments show that generally, if $S$ is a tiny set of generators, then all other tiny objects are retracts of objects in $S$. This method of proof also gives an alternative argument for Proposition 1.46 (the canonical presentation of any functor $C^{\text{op}} \to \text{Set}$ as a colimit of representables) since for $S$ a set of tiny generators, the proof above gives a presentation

$$\text{colim} Y \cong X$$

and the representables form a set of tiny generators by Yoneda’s lemma and the fact that colimits in functor categories are computed pointwise.. Conversely, as an easy generalisation of exercise 2 (i) below one can check that this presentation amounts to an equivalence $\text{Fun}(S^{\text{op}}, \text{Set}) \to C$, so that the hypothesis of admitting a set of tiny generators is a very strong one: Along with cocompleteness it characterises categories of functors to $\text{Set}$ with small domain.

Now clearly, in $\text{Set}$ the only retract of $\ast$ is $\ast$ itself. And any retract diagram in $\text{Fun}(C^{\text{op}}, \text{Set})$ starting at a representable, is determined by an idempotent selfmap of the representing object (which is why the tiny objects of $\text{Fun}(C^{\text{op}}, \text{Set})$ form the idempotent completion of $C$). But there are no idempotent selfmaps in $\Delta$ other than identities (which makes it idempotent complete), which gives the claim about tiny simplicial sets.

Lastly, in a category with a zero object $0$ like $R$–$\text{Mod}$, there are no tiny objects: If $\text{Hom}_C(c, -)$ preserves colimits it has to send an initial object to $\emptyset$, but by definition it has to send a terminal object to $\ast$.

Finally, let me explain what goes wrong in quasi-categories: Given a retract diagram

$$X \xrightarrow{i} Y \xrightarrow{p} X$$

it is not true that $X$ is the coequaliser of $ip$ and $id$: A map from the coequaliser to some object $Z$ now consists of a map $g: Y \to Z$ and a homotopy between $g$ and $gip$. And not every such homotopy arises by taking a map $X \to Z$, precomposing
it with $p$ and using the homotopy $p \pi p \simeq p$. One needs to replace the coequaliser by a larger diagram, that is not finite anymore. That this really cannot be fixed is already apparent in the quasi-category $\text{An}$: Once the theory of limits and colimits in quasicategories is up and running, the argument above shows that the compact objects in $\text{An}$ are precisely the finitely dominated anima, i.e. those Kan complexes which are homotopy retracts of finite simplicial sets. But is a classical theorem of Wall that associated to such an object is an element in a certain algebraic $K$-group, nowadays called Wall’s finiteness obstruction, that vanishes on compact anima. He also gave examples of finitely dominated cell complexes where it does not vanish, and taking their singular set gives a counterexample in the present setting. A similar phenomenon occurs in the derived category of a ring: The compact objects are precisely the perfect complexes, i.e. those which can be represented by bounded complexes, of finitely generated projective modules, whereas the objects build from the compact generator $R$ by finite colimits, always admit bounded representatives consisting of finitely generated free modules. Indeed, projective modules are precisely the idempotent completion of free modules giving one possible proof. □

Exercise 2.

(1) Let $C$ be a cocomplete category with $x \in C$ a tiny generator. Then the functor

$$\text{Hom}_C(x, -) : C \to \text{End}(x)^\text{op}\text{-Set}$$

is an equivalence, where we regard $\text{End}(x)$ as a monoid via composition.

A completely analogous statement holds for categories $\mathcal{A}$ enriched in abelian groups (or in fact any category enriched in a cocomplete category): If the functor $\text{F}_C(x, -) : \mathcal{A} \to \text{Ab}$ preserves colimits and detects isomorphisms, then it promotes to an equivalence $\mathcal{A} \to \text{End}(x)^\text{op}\text{-Mod}$.

(2) Use this assertion to show the categories $R\text{-Mod}$ and $\text{Mat}_n(R)\text{-Mod}$ are equivalent for every ring $R$ and any $n \geq 1$.

Item (2) is a classical theorem of Morita. In his honour rings $R$ and $S$ with equivalent module categories are called Morita equivalent. The analogue of (1) for stable quasi-categories, which is proven the same way too, is known as the Schwede-Shipley theorem.

Solution. The trick is to note that $\text{End}(x)^\text{op}\text{-Set} = \text{Fun}(\mathbb{1}, \text{End}(x)^\text{op}, \text{Set})$. Then by the usual mumbo-jumbo the functor $\text{Hom}_C(x, -)$ is right adjoint to the functor $\text{Fun}(\mathbb{1}, \text{End}(x)^\text{op}, \text{Set}) \to C$, which is the colimit preserving extension of the functor $\mathbb{1} \to C$ that sends the unique object of the source to $x$, with the obvious action on morphisms. We can then apply Lemma I.47: It says that if the left adjoint of an adjunction is fully faithful, and the right adjoint detects isomorphisms, then the functors are inverse equivalences. The second part holds by assumption and for the first we need to check that the adjunction unit $\text{id} \Rightarrow RL$ is an equivalence. In the case at hand both adjoints preserve colimits, so by Theorem I.41 it suffices to check this equivalence for the restrictions to $\mathbb{1} \to \text{Fun}(\mathbb{1}, \text{End}(x)^\text{op}, \text{Set})$.

But by definition the composite of the adjoints sends the unique object on the left to $\text{Hom}_C(x, x)$, i.e. the composite is the Yoneda-embedding, which is also the restriction of the identity.

The second item follows, since, when regarded as a functor $R\text{-Mod} \to \text{Ab}$, the functor $\text{Hom}_R(R^n, -)$ preserves all colimits as it sends $M$ to $M^n$, which is itself a colimit in $R\text{-Mod}$! In fact, the Ab-tiny objects of $R\text{-Mod}$ are precisely the finitely
generated projective modules, by an argument similar to that in the first exercise, since these are exactly the retracts of finite coproducts of the tiny generator $R$. 

**Exercise 3.** Show that a finite-dimensional Kan complex is in fact 0-dimensional, i.e. a constant simplicial set.

This means Kan complexes are either discrete or huge! In particular, this gives another indication that homotopy groups are difficult.

**Solution.** Let $x$ be a non-degenerate $n$-simplex in a Kan complex $K$. If $n > 0$, then we will construct a non-degenerate $n + 1$-simplex in $K$. The idea is simple: There are more horns that can be made from $x$ and degenerations of its faces, than there are degeneracies, so some of these horns have to filled by non-degenerate simplices. For example for $n = 1$, degenerate 2-simplices $y$ are either of the form $d_0 y = d_1 y$ with $d_2 y$ degenerate (i.e. $y = s_0 z$), or $d_1 y = d_2 y$ with $d_0 y$ degenerate (i.e. $y = s_1 z$), but never with $d_1 y$ degenerate, unless all faces are degenerate. But $x$ and $d_0 s_0 x$, for example, define a $(2,0)$-horn that can only be filled by a simplex of the third kind. In general consider the map

$$x \cup s_0 d_0 x : \Delta^n \cup_{\Delta^{n-1}} \Delta^n \to K$$

where the glueing happens via $d_0 : \Delta^{n-1} \to \Delta^n$ on both summands (since $d_0 x = d_0 s_0 d_0 x$). Consider also the inclusion

$$d_1 \cup d_0 : \Delta^n \cup_{\Delta^{n-1}} \Delta^n \to \Delta^{n+1}$$

as two faces of the boundary: This is well defined since $d_1 d_0 = d_0 d_0 : [n-1] \to [n+1]$, with both maps missing 0 and 1. I claim this map is anodyne if $n > 0$. There is then an extension $y : \Delta^{n+1} \to K$ and $y$ is necessarily non-degenerate as if $y = s_1 z$ we find

$$x = d_0 y = d_0 s_0 z = s_{i-1} d_0 z$$

if $i > 0$, or

$$x = d_0 y = d_0 s_0 z = d_1 s_0 z = d_1 y = s_0 d_0 x$$

both of which would need $x$ degenerate.

With all of the machinery we have developed it is easy to see that the inclusion $\Delta^n \cup_{\Delta^{n-1}} \Delta^n \to \Delta^{n+1}$ is anodyne: it is clearly a cofibration since it is readily checked that $d_0 (\Delta^n) \cap d_1 (\Delta^n) = d_0 d_0 \Delta^{n-1}$ and both sides are contractible, so it is a weak homotopy equivalence for $n > 0$. One can also directly exhibit this inclusion or any other as an iterated pushout of horn inclusion. Markus Land does this in 3.16 of his notes, it is even right anodyne.

**Exercise 4.** Show that every left fibration $p : X \to Y$, whose fibres are Kan complexes, determines a canonical functor

$$hY \to \text{hoAn}, \quad y \mapsto p^{-1} (y).$$

Show furthermore, that if $p$ is a Kan fibration, this functor takes values in $\text{core(hoAn)}$. In particular, the fibres over different points in the same path component of $Y$ are equivalent.

As already announced in the lectures, we will soon see that the fibres of every left fibration are Kan complexes, so that this assumption can be removed from the statement.

This result screams for the following generalisation: If $Y$ is a quasi-category and $p : X \to Y$ a left fibration, then $p$ determines a canonical functor

$$Y \to \text{An}, \quad y \mapsto p^{-1} (Y).$$

This is Lurie’s straightening construction, on which the Yoneda lemma for quasi-categories relies (and the raison d’être for left fibrations).
We will only cover this result next term.

Solution. The functor, let’s call it $F_p$, is already defined on objects by the formula above. To define it on morphisms, let $f \in Y_1$ with $d_1(f) = y$ and $d_0(f) = z$ be given and consider the lifting problem

$$
\begin{array}{ccc}
\{0\} \times p^{-1}(y) & \to & X \\
\downarrow & & \\
\Delta^1 \times p^{-1}(y) & \overset{\text{pr}_1}{\to} & \Delta^1 \overset{f}{\to} Y \\
\end{array}
$$

Since the left vertical map is left anodyne by V.2.9, there exists a filler $l$, whose homotopy class (as a filler!) is unique by V.2.26. We define $F_p(f)$ as the homotopy class of the restriction of

$$
l_1: \{1\} \times p^{-1}(y) \to \Delta^1 \times p^{-1}(y) \overset{\text{pr}_1}{\to} X
$$
to codomain $p^{-1}(z)$. We are left to show that this assignment respects compositions and identities. If $f$ is degenerate, then the lower horizontal map in the lifting problem factors through $\Delta^0$ (and not just $\Delta^1$), so a lift is given by

$$
\Delta^1 \times p^{-1}(y) \overset{\text{pr}_1}{\to} p^{-1}(y) \subseteq X,
$$

which evidently yields the identity upon restriction. To see compatibility with composition take $l$ as above and $l': \Delta^1 \times p^{-1}(x) \to X$ fillers for the above diagram for the edges $d_0 H = f$ and $d_2 H = g$ of a 2-simplex $H$ with $d_1(g) = x$, then $l'$ and

$$
\Delta^1 \times p^{-1}(x) \overset{\text{id} \times l'}{\to} \Delta^1 \times p^{-1}(y) \overset{l}{\to} X
$$
glue to a map

$$
\Lambda^2_1 \times p^{-1}(x) \to X
$$
whose restriction to $\{2\} \times p^{-1}(x)$ in the source and $p^{-1}(z)$ in the target is $F_p(f) \circ F_p(g)$. But in the lifting diagram

$$
\begin{array}{ccc}
\Lambda^2_1 \times p^{-1}(x) & \to & X \\
\downarrow & & \\
\Delta^2 \times p^{-1}(x) & \overset{\text{pr}_1 \cup \Delta^2}{\to} & \Delta^2 \overset{H}{\to} Y \\
\end{array}
$$

the left vertical map is again left anodyne by V.2.9 (even inner anodyne by VI.2.2, but we do not need that), so there exists a filler $k$. The composition

$$
\Delta^1 \times p^{-1}(x) \overset{d_1 \times \text{id}}{\to} \Delta^2 \times p^{-1}(x) \overset{k}{\to} X
$$
then satisfies the defining criteria for $F_p(d_1(H))$, so

$$
F_p(d_1(H)) = [(k \circ d_1)_1] = [l_1 \circ l'_1] = [l_1] \circ [l'_1] = F_p(f) \circ F_p(g)
$$
as desired.

If $p$ is a Kan fibration, then to each edge $f \in Y_1$ starting at $y$ and going to $z$ we can also assign a homotopy class of maps $G_p(f): p^{-1}(z) \to p^{-1}(z)$ by using the inclusion $\{1\}$ above (of course this part only needs $p$ to be a right fibration). I claim this always produces an inverse to $F_p(f)$: Just as in the case of compositions, the two lifts $l$ and $l'$ defining $F_p(f)$ and $G_p(f)$ combine into lifting problems

$$
\begin{array}{ccc}
\Lambda^2_0 \times p^{-1}(z) & \overset{\{(l' \circ (d_2 \times \text{id})) \cup (l \circ (d_1 \times l'_1))\}}{\to} & X \\
\downarrow & & \\
\Delta^2 \times p^{-1}(x) & \overset{\text{pr}_1}{\to} & \Delta^2 \overset{s_1 f}{\to} Y \\
\end{array}
$$
The left hand sides are anodyne (left and right, respectively), so the diagrams admit lifts $H$ and $G$. By construction the restriction

$$H \circ (d_0 \times p^{-1}(z)) : \Delta^1 \times p^{-1}(z) \to p^{-1}(z)$$

starts at the identity and ends at $l'_1 \circ l$, and similarly

$$G \circ (d_1 \times p^{-1}(y)) : \Delta^1 \times p^{-1}(y) \to p^{-1}(y)$$

starts at $l_1 \circ l'_1$ and ends at the identity. But both these composites lift degenerate edges, and we checked that above (to see that $F_p$ is compatible with units) that this forces the restriction to the endpoints to be homotopic, so

$$G_p(f) \circ F_p(f) = [l'_1 \circ l_1] = \text{id}$$

and the same for the reverse composition. □