Exercise 1. Let $X$ be a simplicial set that admits inner horn fillers up to dimension 3, and consider $f, g \in X_1$ with $d_1 f = d_0 g$. Show that

$$\pi_0 \text{comp}(f, g) = \ast,$$

where

$$\begin{array}{ccc}
\text{comp}(f, g) & \rightarrow & F(\Delta^2, X) \\
\downarrow & & \downarrow \text{res} \\
\Delta^0 & \overset{(f, g)}\rightarrow & F(\Lambda^2_1, X)
\end{array}$$

is cartesian.

Solution. It is immediate from fillability of $(f, g) : \Lambda^1_2 \rightarrow X$ that $\text{comp}(f, g) \neq \emptyset$. We thus have to show that, given two fillers $H$ and $H'$ for the horn $(f, g) : \Lambda^2_1 \rightarrow X$, there is a prism $I : \Delta^1 \times \Delta^2 \rightarrow X$ whose restrictions to $0 \times \Delta^2$ and $1 \times \Delta^2$ are $H$ and $H'$, respectively, and the restriction to $\Delta^1 \times \Lambda^2_1$ factors as $(f, g) \circ \text{pr}_2$. This is mostly an exercise in indexing: To express the requisite maps we need to give names to the simplices of the prism sensibly: I will simply use chains of vertices, e.g. $(00, 10, 11)$ is a certain non-degenerate 2-simplex in $\Delta^1 \times \Delta^2$ (this is possible, since $\Delta^1 \times \Delta^2$ is the image of a simplicial complex, so all simplices are determined by their vertex chains!).

There are in total 6 non-degenerate 0-simplices, 12 non-degenerate 1-simplices, 10 non-degenerate 2-simplices and 3 non-degenerate 3-simplices.

Now we fill horns as follows: In the 3-simplex $(00, 10, 11, 12)$ the faces $(00, 10, 11)$ and $(10, 11, 12)$ are already given, the former is part of the constant homotopy on $g$ (i.e. it is $s_0 g$), the latter is occupied by $H'$. These are the third and 0th face, respectively. We fill the second one: Its boundary (in order) is given by $(10, 12), (00, 12)$ and $(00, 10)$. The 0th and second face are given by $g$ and $s_0 d_1 g$, which form an inner horn $\Lambda^1_2 \rightarrow X$. Filling it produces the missing side of a map $I$ on the 3-simplex $(00, 10, 11, 12)$. Next up, consider the 3-simplex $(00, 01, 11, 12)$. The boundary conditions for $I$ provide maps on the third and 0th faces $(00, 01, 11)$ and $(01, 11, 12)$, namely $s_1 g$ and $s_0 f$. The first face $(00, 10, 11)$ is also a face of $(00, 10, 11, 12)$, so we have just constructed the map $I$ on it. These three maps combine into a map $\Lambda^2_3 \rightarrow X$, which we can fill, to extend $I$ to the simplex $(00, 01, 11, 12)$. Finally, we extend $I$ over the third non-degenerate 3-simplex $(00, 01, 02, 12)$. The boundary conditions prescribe maps on the third and 0th faces $(00, 01, 02)$ and $(01, 02, 12)$, namely $H$ and $s_1 f$. The second face $(00, 01, 12)$ is part of the simplex $(00, 01, 11, 12)$ so $I$ is already defined there, and the three pieces combine into a map $\Lambda^1_2 \rightarrow X$, which we fill. Thus we have defined $I$ on the all three 3-simplices. Now it is readily observed that their union is the entirety of $\Delta^1 \times \Delta^2$, i.e. every ordered sequence in $(00, 01, 02, 10, 11, 12)$ (with the lexicographic ordering) either contains repeats, which makes it degenerate, or is a subsequence of one of $(00, 10, 11, 12), (00, 01, 02, 12)$ or $(00, 01, 11, 12)$. Geometrically this means that there are no 0, 1 or 2-dimensional bits sticking out of $\Delta^1 \times \Delta^2$). Thus we have produces a map $I : \Delta^1 \times \Delta^2 \rightarrow X$ as desired.
Note also, that this exercise gives another proof of the well-definedness of composition in the homotopy category \( \pi X \) of \( X \): By construction \( \text{comp}(f, g) \subseteq F(\Delta^2, X) \), so we can apply \((d_1)^*: F(\Delta^2, X) \to F(\Delta^1, X)\) to any element in \( \text{comp}(f, g) \) and get a well-defined element in \( \pi_0 F(\Delta^1, X) \) as desired.

In the language now introduced in the lecture (and exercise sheet 6), we have just shown that the inclusion of a prism with a face removed into the full prism, i.e.
\[
\Lambda^2_1 \times \Delta^1 \cup \Delta^2 \times \partial \Delta^1 \subset \Delta^2
\]
is inner 3-anodyne, i.e. in the saturation of the inner horn inclusion up to dimension 3.

**Exercise 2.** Let \( \mathcal{C} \) be a monoidal category with unit \( \mathbb{1} \). Show that \( \text{End}_C(\mathbb{1}, \mathbb{1}) \) is an abelian monoid under composition.

Hint: Show that composition agrees with the operation taking \( f, g \) to \( \eta^{-1} \circ (f \boxtimes g) \circ \eta \). The proof of this statement is often called the Eckmann-Hilton argument.

**Solution.** We follow the hint. First, the suggested operation on \( \text{End}_C(\mathbb{1}, \mathbb{1}) \), let’s abusively call it \( \boxtimes \), is easily verified to be associative and unital (with unit the identity of \( \mathbb{1} \)). Furthermore, we find
\[
(f \boxtimes g) \circ (f' \boxtimes g') = (f \circ f') \boxtimes (g \circ g')
\]
straight from the definitions. But any two binary operations \( \circ \) and \( \boxtimes \) that share a unit and satisfy the above interchange agree and are commutative (this is the Eckmann-Hilton argument):

To see \( f \boxtimes g = f \circ g \) for all \( f, g \in \text{End}_C(\mathbb{1}, \mathbb{1}) \) we compute
\[
f \circ g = (f \boxtimes \text{id}_2) \circ (\text{id}_2 \boxtimes g) = (f \circ \text{id}_2) \boxtimes (\text{id}_2 \circ g) = f \boxtimes g
\]
and to see commutativity we have
\[
f \circ g = (\text{id}_2 \circ f) \circ (g \boxtimes \text{id}_2) = (\text{id}_2 \circ g) \boxtimes (f \circ \text{id}_2) = g \boxtimes f = g \circ f.
\]

**Exercise 3.** Show that the simplicial set of block cellular maps
\[
\mathcal{F}^{bc}_{CW}(X, Y) = \{ \varphi: X \times |\Delta^n| \longrightarrow Y \times |\Delta^n| \mid \varphi \text{ cellular and face preserving} \}
\]
is a Kan complex for every pair of cell complexes \( X \) and \( Y \).

The degeneracies of this simplicial set are constructed in the script, but should not need to feature in your proof.

Also: If you can avoid cellular approximation, I will buy you a beer (or equivalent)!

**Solution.** I still do not know a proof that avoids cellular approximation... so the beer can still be claimed. Here’s one that uses it.

A horn \( \Lambda^n_0 \to \mathcal{F}^{bc}_{CW}(X, Y) \) decodes into a map
\[
\varphi: X \times |\Lambda^n_0| \longrightarrow Y \times |\Lambda^n_0|
\]
that preserves faces and is cellular. To extend it to a map \( X \times |\Delta^n| \longrightarrow Y \times |\Delta^n| \), consider the standard deformation retractions \( r: |\Delta^n| \to |\Lambda^n_0| \) and \( s: |\Delta^n| \to |d_0 \Delta^{n-1}| \), which restrict to inverse homeomorphisms between \( |\Lambda^n_0| \) and \( |d_0 \Delta^{n-1}| \) that are the identity on \( |d_0 \partial \Delta^{n-1}| \). Then
\[
X \times |\Delta^n| \xrightarrow{\varphi} X \times |\Lambda^n_0| \xrightarrow{r} Y \times |\Lambda^n_0| \subseteq Y \times |\Delta^n|
\]
is neither cellular nor face-preserving. To fix that, consider
\[
X \times |d_0 \Delta^{n-1}| \subseteq X \times |\Delta^n| \xrightarrow{\varphi \circ r} Y \times |\Delta^n| \xrightarrow{s} Y \times |d_0 \Delta^{n-1}|.
\]
As both \( r \) and \( s \) restrict to the identity on \( d_i \partial \Delta^{n-1} \), this map agrees with \( \varphi \) on \( d_i \partial \Delta^{n-1} \). In particular, it is cellular and face preserving on \( d_i \partial \Delta^{n-1} \). Let \( \psi : X \times |d_i \partial \Delta^{n-1}| \to Y \times |d_i \partial \Delta^{n-1}| \) denote a relative cellular approximation to it. A homotopy relative to \( d_i \partial \Delta^{n-1} \) witnessing this approximation glues to the constant homotopy of \( \varphi \) to give a homotopy between

\[
(\text{incl} \circ \psi) \cup \varphi : X \times |\Delta^n| \cup X \times |d_i \partial \Delta^{n-1}| = X \times |\partial \Delta^n| \to Y \times |\partial \Delta^n|
\]

and

\[
(s \circ \varphi \circ r) \cup \varphi : X \times |\Delta^n| \cup X \times |d_i \partial \Delta^{n-1}| = X \times |\partial \Delta^n| \to Y \times |\partial \Delta^n|.
\]

But the inclusion of the latter into \( Y \times |\Delta^n| \) is homotopic to \( \varphi \circ r \), which extends over \( X \times |\Delta^n| \). Thus by homotopy extension so does \( \text{incl} \circ \psi \), and any extension solves the original lifting problem by construction. \( \Box \)

**Exercise 4.** Show that the categories \( \mathcal{C}(\Delta_i^n) \) for \( 0 < i < n \) are as described in Lemma IV.33 of the lectures: The evident functor

\[
\mathcal{C}(\Lambda_i^n) \longrightarrow \mathcal{C}(\Delta^n)
\]

is a bijection on objects, and gives an isomorphism of simplicial sets

\[
\mathcal{F}_{\mathcal{C}(\Delta^n)}(j, k) \longrightarrow \mathcal{F}_{\mathcal{C}(\Lambda_i^n)}(j, k) = N([n]_{j,k})
\]

except if \((j, k) = (0, n)\), in which case the analogous map is injective with image consisting of those simplices in \((I_0 \subseteq I_1 \cdots \subseteq I_l) \in N([n]_{0,n}) \) that satisfy

\[
I_0 \neq \{0, n\} \text{ or } I_l \cup \{i\} \neq [n].
\]

Hint: Consider the various inclusions \( d_i : \Delta^{n-1} \to \Lambda_i^n \) for \( i \neq j \). You may use that \( \mathcal{C} : sSet \to \text{Cat}^\Delta \) sends inclusions of simplicial sets to inclusions of simplicially enriched categories. This ultimately follows from pushouts of inclusions of categories being inclusions of categories again, which is easily verified from our construction of colimits in \( \text{Cat} \).

**Solution.** First, let us check that the simplicial subsets described in the exercise, really assemble into a subcategory, let’s call it \( \mathcal{C}_i \), of \( \mathcal{C}(\Delta^n) \): The described sequence of subsets of \( N([n]_{0,n}) \) clearly is closed under doubling and removing entries, so all we need to check is that composition

\[
\mathcal{F}_{\mathcal{C}(\Lambda_j^n)}(j, n) \times \mathcal{F}_{\mathcal{C}(\Delta^n)}(0, j) \to \mathcal{F}_{\mathcal{C}(\Delta^n)}(0, n)
\]

takes values in the prescribed subset for all \( 0 \leq j \leq n \). The cases \( j = 0, n \) are immediate, since \( \mathcal{F}_{\mathcal{C}(\Delta^n)}(j, j) = \Delta^0 \) (even for arbitrary \( j \)), so composition reduces to the identity in this case. For \( 0 < j < n \) we recall that composition is by union of subsets, and for \((I_0 \subseteq \ldots \subseteq I_l) \in N([n]_{0,j})\) and \((J_0 \subseteq \ldots \subseteq J_l) \in N([n]_{j,n})\), we have

\[
j \in I_0 \cup J_0,
\]

so the first condition is satisfied for \((I_0 \cup J_0 \subseteq \ldots \subseteq I_l \cup J_l) \in N([n]_{0,n})\). As it will be useful in a second let us also observe, that any chain \((K_0 \subseteq \ldots \subseteq K_l) \in N([n]_{0,n})\), with \( j \in K_0 \) is obtained from a (unique) pair \( I, J \) as above, by setting

\[
I_j = K_j \cap [0, j] \quad \text{and} \quad J_j = K_j \cap [j, n]
\]

To obtain the claim then, consider the functors

\[
\mathcal{C}(\Delta^{n-1}) \xrightarrow{d_j} \mathcal{C}(\Lambda_i^n) \longrightarrow \mathcal{C}(\Delta^n)
\]

for \( j \neq i \). As mentioned in the hint, I will use that these are injective on both objects and simplicial morphism sets and in particular treat the second as an inclusion (a detailed proof is given in Corollary 1.14 of Joyal’s *Quasi-Categories vs Simplicial categories*). We will first check, that the compositions take values in \( \mathcal{C}_i \). To see this
observe that $d_j$ generally gives an isomorphism of posets $[n - 1] \to [n] \setminus \{j\}$, so the image of the composite map

$$N[n - 1]_{k,l} = F_{\epsilon(\Delta^n - 1)}(k, l) \xrightarrow{d_j} F_{\epsilon(\Delta^n)}(d_jk, d_jl) = N[n]_{d_jk, d_jl}$$

is precisely $N([n] \setminus \{j\})_{d_j, d_j}$. The only non-trivial case to check is $(d_jk, d_jl) = (0, n)$ (that is $(k, l) = (0, n - 1)$ and $0 < j < n$), as otherwise

$$F_{\mathcal{C}}(d_{j}k, d_{j}l) = F_{\epsilon(\Delta^n)}(d_{j}k, d_{j}l).$$

But in this case a chain $(I_0 \subseteq \cdots \subseteq I_l) \in (N[n]_{0,n})_l$ is in the image of $d_j$ if and only if $j \notin I_l$. As $j \neq i$ this implies $I_l \cup \{i\} \neq [n]$, so $(I_0 \subseteq \cdots \subseteq I_l) \in F_{\mathcal{C}}(0, n)_{l}$ as desired.

As $\mathcal{C}$ preserves colimits, we can now compute

$$\mathcal{C}(\Lambda^n_i) = \mathcal{C}\left(\colim_{(\Delta^n, F) \in \mathcal{Y} / \Lambda^n_i} F\right) = \colim_{(\Delta^n, F) \in \mathcal{Y} / \Lambda^n_i} \mathcal{C}(F)$$

using the canonical presentation of every simplicial set as a colimit of simplices from II.6. But every map $F: \Delta^n \to \Lambda^n_i$ factors through $d_j: \Delta^n - 1 \to \Lambda^n_i$ for some $i \neq n$ (by definition of $\Lambda^n_i$). So for each such $F$ the map

$$\mathcal{C}(\Delta^n) \xrightarrow{F} \mathcal{C}(\Lambda^n_i) \to \mathcal{C}(\Delta^n)$$

factors through

$$\mathcal{C}(\Delta^n - 1) \xrightarrow{d_j} \mathcal{C} \subseteq \mathcal{C}(\Delta^n).$$

Thus also the map from the colimit has to factor like that and we obtain that

$$\mathcal{C}(\Lambda^n_i) \subseteq \mathcal{C} \subseteq \mathcal{C}(\Delta^n).$$

It remains to show that the former inclusion is an equality.

To see this, note first that for either $0 \leq k \leq l < j$ or $j < k \leq l \leq n - 1$ we have

$$([n] \setminus \{j\})_{k,l} = [n]_{k,l},$$

so that

$$F_{\epsilon(\Delta^n - 1)}(k, l) \subseteq F_{\epsilon(\Delta^n)}(k, l)$$

is bijective in this case. Applying this with $j = 0, n$ (which is the only time we shall use that $\Lambda^n_i$ is assumed an inner horn) shows that

$$F_{\epsilon(\Lambda^n_i)}(k, l) = F_{\mathcal{C}}(k, l)$$

for $(j, k) \neq (0, n)$.

We are left to consider $F_{\epsilon(\Lambda^n_i)}(0, n)$. As observed above the maps $d_j$ with $0 < j < n$ and $i \neq j$ have image

$$N([n] \setminus \{j\})_{0,n} \subseteq N([n]_{0,n}) = F_{\epsilon(\Delta^n)}(0, n).$$

These nerves precisely account for those simplices $(I_0 \subseteq \cdots \subseteq I_l) \in F_{\mathcal{C}}(0, n)_l$ satisfying $I_l \cup \{i\} \neq [n]$. But any other simplex $(I_0 \subseteq \cdots \subseteq I_l) \in F_{\mathcal{C}}(0, n)_l$ has to satisfy $I_l \neq [0, n]$, say $0, n \neq j \notin I_0$, and we observed in the first paragraph that such simplices lie in the image of composition

$$F_{\epsilon(\Delta^n)}(j, n) \times F_{\epsilon(\Delta^n)}(0, j) \to F_{\epsilon(\Delta^n)}(0, n),$$

so since elements on the left are contained in $\mathcal{C}(\Lambda^n_i)$ so are their composites. $\square$
Let me also mention that for the outer horns, say the 0th, the categories $\mathcal{C}(\Lambda^n_0)$ admit a similar description: Again it has objects $\{0, \ldots, n\}$, and

$$\mathcal{F}_{\mathcal{C}(\Lambda^n_0)}(j,k) \rightarrow \mathcal{F}_{\mathcal{C}(\Delta^n)}(j,k)$$

is bijective unless $(j,k) = (0,n)$ or $(1,n)$, in which case a chain $(I_0 \subseteq \ldots \subseteq I_l) \in N([n]_{j,n})$ is contained in $\mathcal{F}_{\mathcal{C}(\Lambda^n_0)}(j,n)$ if and only if $I_l \cup \{j\} \neq [n]$ or $I_0 \cup \{0,1\} \neq \{0,1,n\}$.

This can be seen just as before: For $(j,k)$ with $k < n$ the inclusion

$$\mathcal{F}_{\mathcal{C}(\Delta^n \setminus (1,0))}(j,k) \rightarrow \mathcal{F}_{\mathcal{C}(\Delta^n)}(j,k)$$

is bijective and for $j > 1$ so is

$$\mathcal{F}_{\mathcal{C}(\Delta^n \setminus (d,0))}(j,k) \rightarrow \mathcal{F}_{\mathcal{C}(\Delta^n)}(j,k).$$

In the remaining cases the chains satisfying the first condition are again the images of $\mathcal{F}_{\mathcal{C}(\Delta^n \setminus 1)}(j,n) \subseteq \mathcal{F}_{\mathcal{C}(\Lambda^n)}(j,n)$ and the remainder are compositions of other simplices.

This time however, the inclusions

$$\mathcal{F}_{\mathcal{C}(\Lambda^n_0)}(j,n) \rightarrow \mathcal{F}_{\mathcal{C}(\Delta^n)}(n,k)$$

are generally anodyne for $j = 1$: For $n = 2$ we find the left hand side empty, for $n = 3$, the left hand side has 2 path components and so on.