Exercise 1. Let $F: C \to D$ be a left Bousfield localisation, i.e. $F$ admits a fully faithful right adjoint $R$. Let $X: I \to D$ be a diagram so that $RX: I \to C$ admits a colimit $c \in C$. Show that $F(c)$ is then a colimit of $X$ in a canonical fashion. In particular, if $C$ is cocomplete, so is $D$.

Deduce that $\text{Cat}$ is cocomplete by means of the nerve functor and use the resulting formula for colimits in $\text{Cat}$ to show that $\text{obj}: \text{Cat} \to \text{Set}$ preserves colimits, whereas $\text{mor}: \text{Cat} \to \text{Set}$ does not.

Another way of showing that $\text{obj}: \text{Cat} \to \text{Set}$ preserves colimits is to observe that it admits not only a left adjoint (namely $\mathbb{D}: \text{Set} \to \text{Cat}$, the discrete category), but also a right one (often called $\mathbb{I}: \text{Set} \to \text{Cat}$, the indiscrete category, with $\mathbb{I}(S)$ consisting of object set $S$ and a single morphism between any two objects).

Similarly, $\text{mor}: \text{Cat} \to \text{Set}$ admits a left adjoint given by $S \mapsto \sum_S \mathbb{B}N$, using the newly found coproducts in $\text{Cat}$ and the functor $\mathbb{B}: \text{Mon} \to \text{Cat}$. Since both the functors $\text{obj}$ and $\text{mor}$ therefore have to preserve limits, there is only one possible thing to try for producing limits in $\text{Cat}$, and indeed, that works (because limits commute!).

Tldr: $\text{Cat}$ is also complete, and limits are as simple as could be.

Solution. There are different ways to rephrase the statement, here is one: Since $F$ is left adjoint, the diagram

$$
\begin{array}{ccc}
I & \xrightarrow{X} & D \\
\downarrow{\text{colim } RX} & & \downarrow{R} \\
\ast & \xrightarrow{F} & C
\end{array}
$$

exhibits $F \circ \text{colim } RX$ as a left Kan extension of $FRX$. Since $FRX \cong X$, $F \circ \text{colim } RX$ is a left Kan extension of $X$ along $I \to \ast$, which was precisely our definition of a colimit.

Consider the adjoint functors $h: \text{sSet} \to \text{Cat}$ and $N: \text{Cat} \to \text{sSet}$. Recall that $N$ is fully faithful. Since $\text{sSet}$ is cocomplete, the previous observation implies that $\text{Cat}$ is cocomplete.

By inspection, $\text{obj} \circ h = (-)_0$ and $(-)_0 \circ N = \text{obj}$. For a diagram $C: I \to \text{Cat}$, we therefore have

$$\text{obj}(\text{colim } C) \cong \text{obj}(h(\text{colim } NC)) \cong (\text{colim } NC)_0 \cong \text{colim } (\text{obj}(C)).$$

On the other hand, consider the pushout of

$$
\begin{array}{ccc}
[0] & \xrightarrow{0\rightarrow 0} & [1] \\
\downarrow{0\rightarrow 1} & & \\
[1] & & 
\end{array}
$$

in $\text{Cat}$. The pushout of the nerves of these categories is $\Lambda^2_1$, and $h(\Lambda^2_1) \cong [2]$. Note that the 1-skeleton of $\Lambda^2_1$ is in bijection with the pushout of the morphism sets. The
category [2] evidently has an additional morphism \((0, 2)\), so \(\text{mor}\) does not preserve colimits.

Given a quasi-category \(\mathcal{C}\), define the core of \(\mathcal{C}\) to be the simplicial subset \(\text{core}(\mathcal{C}) \subseteq \mathcal{C}\) consisting of those simplices in \(\mathcal{C}\), all of whose 1-dimensional faces are equivalences.

**Exercise 2.** Show that \(\text{core}(\mathcal{C})\) is an anima for every quasi-category \(\mathcal{C}\) and that for any ordinary small category \(\mathcal{D}\) there is a canonical natural isomorphism

\[
\text{core}(\mathcal{N}\mathcal{D}) \cong \text{core}(\mathcal{D}).
\]

Deduce, that \(\mathcal{N}\mathcal{D}\) is a Kan complex if and only if \(\mathcal{D}\) is a groupoid.

You do not need to use Joyal’s theorem that all anima are Kan complexes!

**Solution.** Thanks to Bastiaan Cnossen for pointing out that the indices in a previous version of this solution were mostly incorrect, and for providing the correct indices. Any remaining mistakes are still my (=Christoph’s) own.

We need to check that \(\text{core}(\mathcal{C})\) is actually a quasi-category. For \(n \geq 3\), the 1-skeleton of any (inner) horn \(\Lambda^n_\bullet\) equals the 1-skeleton of \(\Delta^n\), so any filler of a map \(\Lambda^n_\bullet \to \text{core}(\mathcal{C}) \to \mathcal{C}\) automatically provides a filler in \(\text{core}(\mathcal{C})\). Hence we only need to consider the case \(n = 2\).

Let \(\Lambda^2_1 \to \text{core}(\mathcal{C})\) be any map. Denote the edges given by \(d_2 \Delta^2\) and \(d_0 \Delta^2\) by \(f\) and \(g\), respectively, and let \(x, y, z\) be the vertices. Choose 2-simplices \(F, G, C\) such that

\[
d_0 F = f, \quad d_1 F = s_0(y), \quad d_0 G = g, \quad d_1 G = s_0(z)\]

\(d_2 C = f\) and \(d_0 C = g\). We need to show that \(d_1 C\) is an equivalence.

Choose also a 2-simplex \(C’\) such that \(d_2 C’ = d_2 F\) and \(d_0 C’ = d_1 C\). We construct several new simplices in \(\mathcal{C}\) by horn-filling. There exists a 3-simplex \(S\) in \(\mathcal{C}\) such that \(d_3 S = F, d_2 S = C’\) and \(d_0 S = C\). Set \(H := d_1 S\). Then there is a 3-simplex \(T\) with \(d_3 T = s_1 d_2 G, d_1 T = G\) and \(d_0 T = H\). Set \(G’ := d_2 T\). Choose a 2-simplex \(I\) with \(d_2 I = d_2 G\) and \(d_0 I = d_2 F\). Finally, there exists a 3-simplex \(U\) such that \(d_3 U = I, d_2 U = G’\) and \(d_0 U = C’\). Consider \(R := d_1 U\). This has faces \(d_0 R = d_1 C\) and \(d_1 R = s_0 z\), so \(d_2 R = d_2 I\) is a right inverse to \(d_1 C\).

In analogous fashion, one shows that there is a 2-simplex \(L\) witnessing that \(d_1 C\) has a left inverse, ie \(d_2 L = d_1 C\) and \(d_1 L = s_0 z\). Then there exists a 3-simplex \(X\) such that \(d_3 X = R, d_1 X = s_0 d_2 R\) and \(d_0 X = L\). Then \(d_2 X\) witnesses that \(d_2 R\) and \(d_0 L\) are homotopic. By Exercise 3 on Sheet 4, it follows that \(d_1 C\) has a two-sided inverse.

Let \(\mathcal{D}\) be a small category. If \(\mathcal{N}\mathcal{D}\) is a Kan complex, there exist for every morphism \(f\) fillers for the horns \(H_L\) and \(H_R\) with \(d_2 H_L = f, d_1 H_L = \text{id}, d_0 H_R = f, d_1 H = \text{id}\). Then \(d_0 H_L\) and \(d_2 H_R\) provide a left and right inverse to \(f\), so \(f\) is an isomorphism.

Conversely, if \(\mathcal{D}\) is a groupoid, the inverse of \(f\) provides a filler for the horns \(H_L\) and \(H_R\), so every edge in \(\mathcal{N}\mathcal{D}\) is an equivalence.

**Exercise 3.** Verify that for a small category \(\mathcal{C}\), we have canonical isomorphisms of simplicial sets

\[
\text{Hom}_{\mathcal{N}\mathcal{C}}(x, y) \cong \mathbb{D}\text{Hom}_{\mathcal{C}}(x, y)
\]

and

\[
\text{comp}_{\mathcal{N}\mathcal{C}}(f, g) \cong \Delta^0
\]

for any \(f, g : x \to y\) in \(\mathcal{C}\).

One strategy is to first produce a natural isomorphism

\[
F(X, \mathcal{N}\mathcal{C}) \leftrightarrow \text{NFun}(hX, \mathcal{C}),
\]
extending Exercise 2 from Sheet 4 (this is part of an enhancement of the functors \( h \) and \( N \) to an enriched adjunction between the simplicially enriched categories \( \text{Cat} \) and \( \text{sSet} \)). For this you will have to first show that \( h \) commutes with finite products. Let me warn you immediately, that \( h \) does not preserve infinite products (where does your argument break?). We will see examples of this in the lecture soon.

**Solution.** We follow the strategy outlined above. First, we show that the homotopy category \( h \) commutes with finite products. Since \((-)_0 = \text{obj} \circ h\), \( h(X \times Y) \) and \( h(X) \times h(Y) \) have isomorphic object sets. Basically by definition, the comparison map \( h(X \times Y) \rightarrow h(X) \times h(Y) \) is also surjective on morphisms. For injectivity, it suffices to observe that if sequences \( f \) and \( f' \) (respectively \( g \) and \( g' \)) are identified in \( h(X) \) (respectively \( h(Y) \)), then \((f, g)\) is identified with \((f', g')\) since we can get from one pair to the other by a finite sequence of “moves”. This was also discussed in the lecture.

Let \( X \) be a simplicial set and let \( \mathcal{C} \) be a category. For every natural number \( n \), we have isomorphisms

\[
F(X, N\mathcal{C})_n = \text{Hom}_{\text{sSet}}(X \times \Delta^n, N\mathcal{C}) \\
\cong \text{Hom}_{\text{Cat}}(h(X \times \Delta^n), \mathcal{C}) \\
\cong \text{Hom}_{\text{Cat}}(h(X) \times h(\Delta^n), \mathcal{C}) \\
\cong \text{Hom}_{\text{Cat}}(h(X) \times N[n], \mathcal{C}) \\
\cong \text{Hom}_{\text{Cat}}(N[n], \text{Fun}(h(X), \mathcal{C})) \\
= N\text{Fun}(h(X), \mathcal{C})_n,
\]

and these are natural in \( n \). So \( F(X, N\mathcal{C}) \cong N\text{Fun}(hX, \mathcal{C}) \).

Let now \( x, y \in \mathcal{C} \). The pullback defining \( \hom_{N\mathcal{C}}(x, y) \) becomes identified with the pullback of

\[
N\text{Fun}([1], \mathcal{C}) \quad \downarrow \\
N[0] \quad \xrightarrow{(x, y)} \quad N\mathcal{C} \times N\mathcal{C}
\]

Since \( N \) is a right adjoint, it preserves limits. So we may compute the pullback in \( \text{Cat} \). The pullback of

\[
\text{Fun}([1], \mathcal{C}) \quad \downarrow \\
[0] \quad \xrightarrow{(x, y)} \quad \mathcal{C} \times \mathcal{C}
\]

has objects the set \( \hom_{\mathcal{C}}(x, y) \) and only identity morphisms, so \( \hom_{N\mathcal{C}}(x, y) \cong \mathcal{D}\hom_{\mathcal{C}}(x, y) \).

Let \( f, g \) be composable maps. We wish to argue similarly for the pullback

\[
\text{comp}_{N\mathcal{C}}(f, g) \quad \rightarrow \quad F(\Lambda^2_1, N\mathcal{C}) \quad \downarrow \\
\Delta^0 \quad \xrightarrow{(f, g)} \quad N\mathcal{C} \times N\mathcal{C}
\]

Since \( h\Lambda^2_1 \cong [2] \), this translates to computing the pullback of

\[
\text{Fun}([2], \mathcal{C}) \quad \downarrow \\
[0] \quad \xrightarrow{(f, g)} \quad \mathcal{C} \times \mathcal{C}
\]
in Cat. By inspection, this pullback has precisely one object and one morphism.

**Exercise 4.** Let \( L : C \rightleftarrows D : R \) be adjoint functors with \( C \) and \( D \) monoidal categories. Produce a bijection between lax monoidal structures on \( R \) and oplax monoidal structures on \( L \), such that a natural transformation between lax monoidal right adjoints is monoidal if and only if the corresponding natural transformation between oplax monoidal left adjoints is monoidal.

**Solution.** This time, we only sketch the proof. Let \((C, \otimes)\) and \((D, \boxtimes)\) denote the monoidal structures. Define

\[
\text{lax}(R) := \{ (\tau : \otimes \circ (R \times R) \Rightarrow R \circ \boxtimes, e : 1_C \Rightarrow R(1_D)) \text{ lax structure} \}
\]

and

\[
\text{oplax}(L) := \{ (\eta : L \circ \otimes \Rightarrow \boxtimes \circ (L \times L), k : L(1_C) \Rightarrow 1_D) \text{ oplax structure} \}.
\]

Let \( u \) and \( c \) denote the unit and counit of the adjunction. Then there are maps

\[
o : \text{lax}(R) \to \text{oplax}(L)
\]

\[
(\tau, e) \mapsto (o(\tau) := c \boxtimes (L \times L) \circ L \tau (L \times L) \circ L \otimes (u \times u), o(e) := c \circ L e),
\]

\[
\ell : \text{oplax}(L) \to \text{lax}(R)
\]

\[
(\eta, k) \mapsto (\ell(\eta) := R \boxtimes (c \times c) \circ R \eta (R \times R) \circ u \otimes (R \times R), \ell(k) := R \tau \circ u).
\]

The verification that \( o(\tau, e) \) and \( \ell(\eta, k) \) are (op)lax structures consists of writing out huge commutative diagrams, as always.

Now let us check that \( \ell \circ o = \text{id} \). By naturality and the triangle identities, we have

\[
\ell(o(\tau))
\]

\[
= R \boxtimes (c \times c) \circ R c \boxtimes (LR \times LR) \circ RL \tau (LR \times LR) \circ R \otimes (u R \times u R) \circ u \otimes (R \times R)
\]

\[
= R c \boxtimes R L \tau (LR \times LR) \circ R \otimes (u R \times u R) \circ u \otimes (R \times R)
\]

\[
= R c \boxtimes R \tau \circ u \otimes (R \times R)
\]

\[
= \tau
\]

and

\[
\ell(o(e)) = R L e \circ R c \circ u = R c \circ u R \circ e = e.
\]

The equality \( o \circ \ell = \text{id} \) is completely analogous. So \( \ell \) and \( o \) are mutually inverse bijections.

Let now \((L, R)\) and \((L', R')\) be adjoint pairs \( C \rightleftarrows D \). Denote the corresponding (co)units by \( b, u, c' \) and \( u' \) respectively. Let \( \alpha : (R, \tau) \Rightarrow (R', \tau') \) be a monoidal transformation. Then the corresponding natural transformation \( L' \Rightarrow L \) is given by \( \beta := L' R' L \circ L' a L \circ L' u \). Draw more diagrams to see that \( \beta \) is also monoidal, and that the converse holds.