**Exercise 1.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. Then $F$ is an isofibration if and only if $N \pi F : N \pi \mathcal{C} \to N \pi \mathcal{D}$ is an isofibration.

**Solution.** Suppose that $F$ is an isofibration and consider a lifting problem

$$
\begin{array}{ccc}
\{0\} & \xrightarrow{x} & \text{core } N \pi \mathcal{C} \\
\downarrow & & \downarrow \text{core } N \pi F \\
\Delta^1 & \longrightarrow & \text{core } N \pi \mathcal{D}
\end{array}
$$

Since $\pi$ and $N$ are adjoint, this is equivalent to a lifting problem

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & \text{core}(\pi \mathcal{C}) \\
\downarrow & & \downarrow \text{core } \pi F \\
[1] & \longrightarrow & \text{core}(\pi \mathcal{D})
\end{array}
$$

Choosing any representative of the functor $[1] \to \text{core}(\pi \mathcal{D})$ produces an analogous lifting problem for $F$ which has a solution by assumption and induces a solution to the given problem.

Conversely, consider a lifting problem

$$
\begin{array}{ccc}
\{0\} & \xrightarrow{x} & \text{core } \mathcal{C} \\
\downarrow & & \downarrow \text{core } F \\
\Delta^1 & \xrightarrow{f} & \text{core } \mathcal{D}
\end{array}
$$

This induces an analogous lifting problem for $N \pi F$ which has a solution $g'$ by assumption. Then $F(g')$ is homotopic to $f$ via some homotopy $h$, so we obtain a lifting problem

$$
\begin{array}{ccc}
\Lambda^2_1 & \xrightarrow{(g', \text{id})} & \text{core } \mathcal{C} \\
\downarrow & & \downarrow \text{core } (F) \\
\Delta^2 & \xrightarrow{h} & \text{core } \mathcal{D}
\end{array}
$$
which also has a solution since $\text{core}(F')$ is an inner fibration. Restricting to $d_1$ yields a solution to the original problem. □

Define the class of *Joyal fibrations* as the class of those maps which have the RLP with respect to all morphisms which are cofibrations and Joyal equivalences.

**Exercise 2.** Every Joyal fibration between quasicategories which is a Joyal equivalence is a trivial fibration.

**Solution.** By the small object argument, we can factor $p$ into a cofibration $i: C \to D'$ followed by a trivial fibration $q: D' \to D$. Since $q$ is a Joyal equivalence, it follows by two-out-of-three that $i$ is also a Joyal equivalence. Consequently, the lifting problem

$$
\begin{array}{ccc}
C & \xrightarrow{id_C} & C \\
\downarrow^i & & \downarrow^p \\
D' & \xrightarrow{q} & D
\end{array}
$$

has a solution $s: D' \to C$. These data provide a retraction diagram

$$
\begin{array}{ccc}
C & \xrightarrow{i} & D' & \xrightarrow{s} & C \\
\downarrow^p & & \downarrow^q & & \downarrow^p \\
D & \xrightarrow{id_D} & D & \xrightarrow{id_D} & D
\end{array}
$$

Being a retract of a trivial fibration, $p$ is a trivial fibration. □

**Exercise 3.**

(1) Every Joyal fibration is an isofibration (ie has the RLP with respect to $J$-anodyne maps).

(2) Every isofibration between quasicategories is a Joyal fibration.

(3) An isofibration between simplicial sets need not be a Joyal fibration. Similarly, a cofibration that is also a Joyal equivalence need not be $J$-anodyne.

**Hint for (3):** Take another look at exercise 4 on sheet 14.

**Solution.** (1) By exercise 3 on sheet 14, every $J$-anodyne map is a cofibration and a Joyal equivalence. Since isofibrations are precisely the maps with the RLP with respect to $J$-anodynes, it follows that every Joyal fibration is an isofibration.
(2) Let $p: C \to D$ be an isofibration between quasicategories and let $i: A \to B$ be a cofibration and a Joyal equivalence. Any lifting problem

$$
\begin{array}{c}
A \\
\downarrow \\
B
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow^p \\
\longrightarrow
\end{array} \quad \begin{array}{c}
C \\
\downarrow \\
D
\end{array}
$$

translates to a lifting problem

$$
\begin{array}{c}
\emptyset \\
\downarrow
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Fun}(B, C) \\
\downarrow^q
\end{array} \quad \begin{array}{c}
\text{Fun}(A, C) \times ^{\text{Fun}(A,D)} \text{Fun}(B, D) \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Fun}(B, D)
\end{array}
$$

so we only have to show that $q$ is surjective on 0-simplices. We will show that it is a trivial fibration.

Consider the commutative diagram

$$
\begin{array}{c}
\text{Fun}(B, C) \\
\downarrow^{i^*} \\
\text{Fun}(A, C)
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Fun}(A, C) \times ^{\text{Fun}(A,D)} \text{Fun}(B, D) \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{Fun}(B, D)
\end{array}
$$

The map $i^*$ is an isofibration (by VII.11) and a Joyal equivalence, so it is a trivial fibration by VII.15. Consequently, the second vertical map is also a trivial fibration because the displayed square is a pullback. In particular, it is a trivial fibration and thus a Joyal equivalence. Since $i$ is assumed to be a Joyal equivalence, $q$ is a Joyal equivalence by two-out-of-three. Since $q$ is also an isofibration by VII.11, it follows again from VII.15 that $q$ is a trivial fibration.

(3) Let $i: \Delta^1 \to X$ be Campbell’s map (from exercise 4 on sheet 14). There are only two maps $J \to X$: $X$ comes with a natural map $X \to \Delta^1$ induced by $s_0$. Its fibres over the vertices of $\Delta^1$ are easily checked to consist of a point each. But every map $J \to \Delta^1$ is constant (it is the same as a functor from the free isomorphism to [1] after all), so in particular, any map $J \to X$ has to factor through a fibre of $X \to \Delta^1$ and is therefore constant. As $i$ is an inner fibration (we verified this in the proof of exercise 4, sheet 14) and clearly bijective on vertices, it follows that $i$ is an isofibration by exercise 4, sheet 12. If $i$ were a Joyal fibration, then as $i$ is both a cofibration and a Joyal equivalence
(again by exercise 4, sheet 14) the lifting problem
\[
\begin{array}{ccc}
\Delta^1 & \xrightarrow{\text{id}} & \Delta^1 \\
i & \downarrow & \downarrow i \\
X & \xrightarrow{\text{id}} & X
\end{array}
\]
would have a solution, forcing \(i\) to be an isomorphism, which is evidently false. Similarly, if \(i\) were \(J\)-anodyne, this diagram would have a solution.

\[\square\]

Let \(\text{Cat}_{(2,1)}\) denote the groupoid-enriched category of small categories (ie mapping objects are the maximal groupoids \(\text{core} \text{Fun}(\mathcal{C}, \mathcal{D})\) in the functor category \(\text{Fun}(\mathcal{C}, \mathcal{D})\)). Since the nerve functor \(N: \text{Cat} \to \text{sSet}\) is monoidal, there is an induced change-of-enrichment functor \(N_*: \text{Cat} \to \text{sSet}\). Abbreviate \(N_* N_* \text{Cat}_{(2,1)}\) to \(N_* \text{Cat}_{(2,1)}\).

**Exercise 4.**

1. Show that the nerve induces an enriched functor \(N_* \text{Cat}_{(2,1)} \to \text{qCat}\) to the Kan-enriched category of quasicategories.
2. The induced functor \(N_* \text{Cat}_{(2,1)} \to \text{Cat}_\infty\) is fully faithful. Observe that this functor induces an equivalence to the full subcategory of those quasicategories with homotopically discrete Hom-anima.
3. Show that this functor induces an equivalence between the full subcategory \(N_* \text{Grpd}\) of groupoids and the full subcategory of anima which are 1-types (ie have \(\pi_i = 0\) for all \(i \geq 2\)).
4. Show that this functor does not induce an equivalence between the (quasi)category of groups \(N \text{Grp}\) and the full subcategory of connected 1-types.

**Solution.** (1) We claim that the following assignments define an enriched functor \(N_* \text{Cat}_{(2,1)} \to \text{qCat}\):

i) On objects, send a (small) category \(\mathcal{C}\) to its nerve \(N \mathcal{C}\). Note that \(N \mathcal{C}\) is a quasicategory.

ii) For any two categories \(\mathcal{C}\) and \(\mathcal{D}\), let
\[
F_{\mathcal{C}, \mathcal{D}}: \text{core} \text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{core} \text{Fun}(N \mathcal{C}, N \mathcal{D})
\]
be given by the canonical isomorphism
\[
\text{core} \text{Fun}(\mathcal{C}, \mathcal{D}) \cong \text{core} N \text{Fun}(\mathcal{C}, \mathcal{D}) \cong \text{Fun}(N \mathcal{C}, N \mathcal{D}),
\]
where the second isomorphism comes from exercise 2 on sheet 4.
Then $F_{C,D}$ has the correct source and target. It is straightforward to check that the diagrams

\[
\begin{CD}
\text{N core Fun}(D, E) \times \text{N core Fun}(C, D) @> F_{D,E} \times F_{C,D} >> \text{core Fun}(N D, N E) \times \text{core Fun}(N C, N D) \\
\downarrow @. \downarrow \\
\text{N core Fun}(C, E) @> F_{C,E} >> \text{core Fun}(N C, N E)
\end{CD}
\]

and

\[
\begin{CD}
\Delta^0 @< F_{C,C} << \text{N core Fun}(C, C) \\
\downarrow @. \downarrow \\
\text{core Fun}(N C, N C)
\end{CD}
\]

commute.

(2) By VII.19, the induced maps

\[
\text{Hom}_{\text{Cat}_{(2,1),N}}(C, D) \rightarrow \text{Hom}_{\text{Cat}_\infty}(N C, N D)
\]

are equivalences since $F_{C,D}$ is an equivalence (it is even an isomorphism). The additional observation follows directly from VII.14 together with Theorem F.

(3) Since nerves of groupoids are Kan complexes and Kan complexes are the same as anima by Theorem C, the restriction of $N^c \text{Cat}_{(2,1)} \rightarrow \text{Cat}_\infty$ to the full subcategory of groupoids induces an equivalence to the full subcategory spanned by all anima with homotopically discrete Hom-anima. For $X$ an anima, note that $\text{Hom}_X(x, y) \simeq \text{Hom}_X(x, x)$ unless $\text{Hom}_X(x, y)$ is empty. Consequently, it suffices to consider endomorphism anima $\text{Hom}_X(x, x)$ for all $x \in X$. After (or during the proof of) Lemma VII.10, we observed that $\text{Hom}_X(x, x) \simeq \Omega_x X$. So $\text{Hom}_X(x, x)$ is homtopically discrete if and only if $\pi_i(X, x) = 0$ for all $x \in X$ and $i \geq 2$.

(4) Note that the full subcategory of Grpd spanned by all groups is equivalent to the full subcategory spanned by all connected groupoids by exercise 4 i) on sheet 1. Consequently, the full subcategory of $N^c \text{Grpd}$ spanned by all groups is equivalent to the full subcategory $\text{An}$ spanned by the connected 1-types. Thus, it suffices to show that the canonical functor $\text{Grp} \rightarrow \text{Grpd}$ is not fully faithful (as groupoid-enriched categories). In fact, $\text{Grp}$ is an ordinary category, while $\text{Grpd}$ has non-trivial 2-morphisms: for $G$ a group and $g \in G$, the morphism $g : \ast \rightarrow \ast$ defines a natural isomorphism $\text{id}_G \Rightarrow c_g$ to the conjugation with $g$ as we saw in exercise 2, sheet 4. Thus the canonical map
$\text{Hom}_{\text{Grp}}(G, G) \rightarrow \mathcal{F}_{\text{Grpd}}(G, G)$ is not injective on $\pi_0$ whenever $G$ is non-abelian.

This accounts for the fact that we did not fix base points of 1-types; the quasicategory of pointed 1-types (How would you define this?) is equivalent to $N\text{Grp}$. □