Exercise 1. Let $F : C \to D$ be an inner fibration between quasicategories. Show that the induced map

$$\text{Hom}_C(x, y) \to \text{Hom}_D(Fx, Fy)$$

is a Kan fibration for all $x, y \in C$.

Solution. For all $x, y \in C$, the commutative square

$$\begin{array}{ccc}
\Delta^0 \times \text{Fun}(\partial \Delta^1, C) & \longrightarrow & \text{Fun}(\partial \Delta^1, C) \\
\downarrow & & \downarrow \\
\Delta^0 \times \text{Fun}(\partial \Delta^1, D) & \longrightarrow & \text{Fun}(\partial \Delta^1, C) \times \text{Fun}(\partial \Delta^1, D)
\end{array}$$

is a pullback since limits commute with each other, where the map $\Delta^0 \to \text{Fun}(\partial \Delta^1, C) \cong C \times C$ classifies $x$ and $y$, and $\Delta^0 \to \text{Fun}(\partial \Delta^1, D)$ classifies $Fx$ and $Fy$. Now observe that this pullback is isomorphic to the diagram

$$\begin{array}{ccc}
\text{Hom}_C(x, y) & \longrightarrow & \text{Fun}(\Delta^1, C) \\
\downarrow & & \downarrow \\
\text{Hom}_D(Fx, Fy) & \longrightarrow & \text{Fun}(\partial \Delta^1, C) \times \text{Fun}(\partial \Delta^1, D)
\end{array}$$

The right vertical map is an isofibration by VII.11 since $F$ is an inner fibration and $\partial \Delta^1 \to \Delta^1$ a cofibration which is bijective on 0-simplices. Therefore, $\text{Hom}_C(x, y) \to \text{Hom}_D(Fx, Fy)$ is an isofibration since by exercise 3 on sheet 12, these are characterised by a right lifting property and thus stable under pullback. Since an isofibration of Kan complexes is a Kan fibration by the same exercise, the claim follows.

Note that Fabian once mentioned during the lectures, that an inner fibration between Kan complexes is automatically a Kan fibration: This is nonsense! By Joyal’s theorem all outer horns of dimension at least 2 admit lifts in this case, but the outer 1-horns do not (this is precisely what being an isofibration means in this case). For a concrete counterexample consider $\Delta^0 \to J$. \hfill \qed
Exercise 2. Let $F: \mathcal{C} \to \mathcal{D}$ be a fully faithful functor between quasicategories. Show that the induced functor

$$F_*: \text{Fun}(X, \mathcal{C}) \to \text{Fun}(X, \mathcal{D})$$

is fully faithful for every simplicial set $X$.

Solution. Let $\mathcal{D}_F$ denote the essential image of $F$, ie the strict full subcategory of $\mathcal{D}$ spanned by objects which are equivalent to objects in the image of $F$. Then $F$ induces an essentially surjective and fully faithful functor $F': \mathcal{C} \to \mathcal{D}_F$. By Theorem F, $F'$ is a Joyal equivalence of quasicategories. Consequently, the induced functor

$$F'_*: \text{Fun}(X, \mathcal{C}) \to \text{Fun}(X, \mathcal{D}_F)$$

is a Joyal equivalence; in particular, it is fully faithful.

Moreover, $\text{Fun}(X, \mathcal{D}_F)$ is a strict full subcategory of $\text{Fun}(X, \mathcal{D})$ since it fits into a pullback square

$$\begin{array}{ccc}
\text{Fun}(X, \mathcal{D}_F) & \longrightarrow & \text{Fun}(X, \mathcal{D}) \\
\downarrow & & \downarrow \\
\mathcal{A} & \longrightarrow & \pi \text{Fun}(X, \mathcal{D})
\end{array}$$

where $\mathcal{A}$ denotes the full subcategory of $\pi \text{Fun}(X, \mathcal{D})$ spanned by those functors which only take values in $\mathcal{D}_F$.

It follows that $F_*$ is the composition of two fully faithful functors. \qed

Let $J$ denote the nerve of the free isomorphism. Call a map of simplicial sets $J$-anodyne if it lies in the saturated class generated by \{0\} \to J and all inner horn inclusions.

Exercise 3.

(1) Every $J$-anodyne map is a cofibration and a Joyal equivalence.

(2) If $i: A \to \mathcal{C}$ is a cofibration and a Joyal equivalence with $\mathcal{C}$ a quasicategory, then $i$ is $J$-anodyne.

Solution. (1) It suffices to show that the class of cofibrations which are Joyal equivalences form a saturated class: Since all inner anodyne maps and \{0\} \to J are both cofibrations and Joyal equivalences, this implies the claim.

Retracts and coproducts of cofibrations/Joyal equivalences are evidently cofibrations/Joyal equivalences. Given a pushout diagram

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow_i & & \downarrow_j \\
X & \longrightarrow & Y
\end{array}$$
with $i$ both a cofibration and a Joyal equivalence, $j$ is also a cofibration. Applying $\text{Fun}(-, \mathcal{C})$ with $\mathcal{C}$ a quasicategory yields a pullback square

$$
\begin{array}{ccc}
\text{Fun}(Y, \mathcal{C}) & \xrightarrow{j^*} & \text{Fun}(B, \mathcal{C}) \\
\downarrow s^* & & \downarrow f^* \\
\text{Fun}(X, \mathcal{C}) & \xrightarrow{i^*} & \text{Fun}(A, \mathcal{C})
\end{array}
$$

By VII.11, $i^*$ is an isofibration. By assumption it is also a Joyal equivalence, so it is a trivial fibration by VII.15. Consequently, $j^*$ is also a trivial fibration. In particular, it is a Joyal equivalence. Since $\mathcal{C}$ was arbitrary, this shows that $j$ is a Joyal equivalence.

Let now $X(0) \xrightarrow{i(1)} X(1) \xrightarrow{i(2)} \ldots$ be given such that $i(n)$ is a cofibration and a Joyal equivalence for all $n$. Then we obtain for every quasicategory $\mathcal{C}$ a tower

$$
\ldots \xrightarrow{i(2)^*} \text{Fun}(X(1), \mathcal{C}) \xrightarrow{i(1)^*} \text{Fun}(X(0), \mathcal{C})
$$

in which all structure maps are isofibrations and Joyal equivalences, hence trivial fibrations. It follows easily that the induced map

$$
\lim_n \text{Fun}(X(n), \mathcal{C}) \cong \text{Fun}(\operatorname{colim}_n X(n), \mathcal{C}) \rightarrow \text{Fun}(X(0), \mathcal{C})
$$

is a trivial fibration. So $X(0) \rightarrow \operatorname{colim}_n X(n)$ is also a Joyal equivalence.

(2) Apply the small object argument to factor $i$ into a $J$-anodyne map $j: A \rightarrow \mathcal{C}'$ followed by an isofibration between quasicategories $p: \mathcal{C}' \rightarrow \mathcal{C}$. Since $J$-anodyne maps are Joyal equivalences as we have just shown, $p$ is also a Joyal equivalence by two-out-of-three. By VII.15, $p$ is a trivial Kan fibration. Consequently, the lifting problem

$$
\begin{array}{ccc}
A & \xrightarrow{j} & \mathcal{C}' \\
\downarrow i & & \downarrow p \\
\mathcal{C} & \xrightarrow{\text{id}_\mathcal{C}} & \mathcal{C}
\end{array}
$$

has a solution $s: \mathcal{C} \rightarrow \mathcal{C}'$. These data provide a retraction diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\text{id}_A} & A \\
\downarrow i & & \downarrow j & \downarrow i \\
\mathcal{C} & \xrightarrow{s} & \mathcal{C}' & \xrightarrow{p} \mathcal{C}
\end{array}
$$

Being a retract of a $J$-anodyne map, $i$ is $J$-anodyne. \hfill \square

Exercise 4.
(1) Suppose that \( i: A \to C \) is a cofibration, a Joyal equivalence and a bijection on 0-simplices, and that \( C \) is a quasicategory. Then \( i \) is inner anodyne.

(2) Show that this statement is wrong if \( C \) is not assumed to be a quasicategory.

Hint: Let \( X \) be the simplicial set defined by the pushout

\[
\begin{array}{ccc}
\Delta^1 & \xrightarrow{d_2} & \Delta^2 \\
\downarrow & & \downarrow p \\
\Delta^0 & \xrightarrow{d_1} & X
\end{array}
\]

and consider the composite map \( i: \Delta^1 \xrightarrow{d_1} \Delta^2 \xrightarrow{p} X \).

Proof. (1) Use the small object argument to factor \( i \) into an inner anodyne map \( j: A \to C' \) followed by an inner fibration \( p: C' \to C \). Since \( C \) is a quasicategory, so is \( C' \). Since \( A \to C \) is bijective on 0-simplices, as are all inner anodyne maps, also \( p \) is a a bijection on 0-simplices and similarly \( p \) is a Joyal equivalence by two-out-of-three. We will show that \( i \) is a retract of \( j \). As before this follows from \( p \) being a trivial fibration as then the lifting problem

\[
\begin{array}{ccc}
A & \xrightarrow{j} & C' \\
\downarrow i & & \downarrow p \\
C & \xrightarrow{id_C} & C
\end{array}
\]

has a solution \( s: C \to C' \) and these data provide a retraction diagram

\[
\begin{array}{ccc}
A & \xrightarrow{id_A} & A \\
\downarrow i & & \downarrow j \\
C & \xrightarrow{s} & C' & \xrightarrow{p} & C.
\end{array}
\]

To see that \( p \) is a trivial fibration we employ VII.15 again. We thus need to show that \( p \) isofibration so consider a lifting problem

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{x} & \text{core}C' \\
\downarrow & & \downarrow \text{core}(p) \\
\Delta^1 & \xrightarrow{g: px \to y} & \text{core}C
\end{array}
\]

The image of \( y \) in \( C \) has a preimage \( x' \) since \( p \) is surjective on 0-simplices. Since \( \text{core}(p) \) is fully faithful by VII.9, there exists a morphism \( f': x \to \)
$x'$ such that $f^h \simeq g$. This gives rise to a lifting problem

$$
\begin{array}{ccc}
\Lambda^2_1 & \xrightarrow{(f,\text{id})} & C' \\
\downarrow & & \downarrow p \\
\Delta^2 & \xrightarrow{h} & C
\end{array}
$$

Since $p$ is an inner fibration, there exists a solution. It follows that $p$ is an isofibration.

(2) The image of $i$ on 0-simplices consists of 0 and 2. Since 0 and 1 are identified in $X$, $i$ is bijective on 0-simplices.

Moreover, $\Delta^1$ has only one more non-degenerate simplex $0 \leq 1$ which is sent to $0 \leq 2$, a non-degenerate simplex in $X$. So $i$ is a cofibration.

Now observe that $X$ can also be written as a pushout

$$
\begin{array}{ccc}
\Lambda^2_1 & \longrightarrow & \Delta^2 \\
\downarrow & & \downarrow p \\
\Delta^1 & \xrightarrow{j} & X
\end{array}
$$

Note that $j$ is inner anodyne by definition, and that it is given by the composition $\Delta^1 \xrightarrow{d_0} \Delta^2 \xrightarrow{p} X$. The map $s_0: \Delta^2 \to \Delta^1$ induces a map $r: X \to \Delta^1$ which is a retraction to both $i$ and $j$. By two-out-of-three, $r$ is a Joyal equivalence and therefore so is $i$.

Finally, one can show that $i$ is also an inner fibration: there is a pullback square

$$
\begin{array}{ccc}
\Lambda^2_0 & \longrightarrow & \Delta^1 \\
\downarrow & & \downarrow i \\
\Delta^2 & \xrightarrow{p} & X
\end{array}
$$

in which the left vertical map is the canonical inclusion. Note that this inclusion is an inner fibration by exercise 1(2) on sheet 11 since $\Delta^2$ is the nerve of a category.

Given a lifting problem

$$
\begin{array}{ccc}
\Lambda^n_k & \longrightarrow & \Delta^1 \\
\downarrow & & \downarrow i \\
\Delta^n & \xrightarrow{x} & X
\end{array}
$$

the map $x$ factors as $px'$ since $p$ is an epimorphism. Then by the universal property of the pullback there is an induced map $\overline{x}': \Lambda^n_k \to \Lambda^2_0$. 
lifting $x'$. Thus, we have a lifting problem

\[
\begin{array}{ccc}
\Lambda^n_k & \xrightarrow{\pi} & \Lambda^2_0 \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{x'} & \Delta^2
\end{array}
\]

which has a solution, and this solution also solves the original lifting problem.

Suppose $i$ is inner anodyne. Since $i$ is also an inner fibration, the lifting problem

\[
\begin{array}{ccc}
\Delta^1 & \xrightarrow{id} & \Delta^1 \\
\downarrow i & & \downarrow i \\
X & \xrightarrow{id} & X
\end{array}
\]

has a solution which shows that $f$ has both a left and a right inverse. So $f$ is an isomorphism, which is obviously false.

\qed