Exercise 1. Show that the saturation of the map \( \emptyset \to * \) in \( \text{Set} \) is precisely given by the injections. Dually, show that the saturation of the map \( \{0, 1\} \to * \) is given by the surjections.

What maps admit right lifts against these classes?

In particular, you are to show that injections and surjections form saturated classes, as was mentioned without proof in the lectures.

Exercise 2. Let \( X \subseteq Y \) be an inclusion of simplicial sets. Construct a relative skeletal filtration on \( Y \), i.e. show that there exist simplicial subsets \( X \subseteq Y_i \subseteq Y \), such that the non-degenerate simplices of \( Y_i \) are precisely the non-degenerate simplices of \( X \) together with all non-degenerate simplices of \( Y \) of dimension at most \( i \).

Deduce that the saturated class spanned by the boundary inclusions \( \partial \Delta^n \to \Delta^n \) in \( \text{sSet} \) is exactly given by the cofibrations.

A map which admits right lifts against all boundary inclusion is called a trivial Kan fibration. As horn inclusions are injections, the exercise implies that these maps really are Kan fibrations. We will study them further in the next few weeks.

Exercise 3. Let \( C \) be a cocomplete category that admit finite products and furthermore that these commute with colimits (e.g. \( C \) is cocomplete and cartesian closed). Let furthermore \( \Sigma \subseteq \text{mor}(C) \) be saturated and \( j: X \to Y \) a morphism in \( C \). Show that

\[
\{ i \in \text{mor}(C) \mid i \boxtimes j \in \Sigma \}
\]

is saturated again.

A weaker statement was announced as an exercise during the lecture, which caused a gap in the proof that \( i \boxtimes j \) is anodyne if \( j \) is injective and \( i \) anodyne. The present formulation of the exercise closes that gap!

Denote by \( \text{IH} \) the set of inner horn inclusions \( \Lambda_i^n \to \Delta^n \) and by \( S \) the set of spine inclusions \( I_n \to \Delta_n \).

Exercise 4. Show that there are strict containments

\[
\text{sat}(S) \subset \text{sat}(\text{IH}) \subset \text{sat}(H),
\]

where the last class is that of anodyne extensions.
Hint: To see that the first containment is strict, consider for example the map from the pushout of
\[ \Delta^2 \leftarrow \Lambda^2_1 \rightarrow \Delta^2. \]
to $\Delta^0$.

Elements in the class sat(IH) are known as \textit{inner anodyne extensions} and maps which admit right lifts against them are known as \textit{inner fibrations}; these classes of maps are to quasi-categories what anodyne extensions and Kan fibrations are to Kan complexes and will thus be very important later in the lecture. For example, essentially by definition $C$ is a quasi-category if and only if $C \rightarrow \Delta^0$ is an inner fibration. The inner anodyne extensions do not, however, admit a sensible description in terms of cylinder inclusions and are somewhat more mysterious than their left and right counterparts.

Let me also mention that the inclusions
\[ \text{sat(IH)} \subset \text{sat(RH)} \subset \text{sat(H)} \]
are also strict as we will see later.

And now that the exercise sheet does not fit onto a single page anymore, let me make another comment: In his notes Markus Land calls a simplicial set $C$ a \textit{composer}, if it admits spine fillers, or in other words if the map $C \rightarrow \Delta^0$ admits right lifts with respect to sat(S). The pushout from the hint is therefore an explicit composer which is not a quasi-category (he takes a more bus-driver route to producing such an example).