Let $\mathcal{C}$ be a category admitting finite products. It is called cartesian closed if the functor $- \times X: \mathcal{C} \to \mathcal{C}$ admits a right adjoint, which we will denote by $F(X, -): \mathcal{C} \to \mathcal{C}$. In particular, in this case the functor $- \times X$ commutes with colimits.

**Exercise 1.** Show that in a cartesian closed category the association $X \mapsto F(X, -)$ canonically extends to a functor $\mathcal{C} \to \text{Fun}(\mathcal{C}, \mathcal{C})$ and that there is a natural isomorphism

$$F(- \times -, -) \cong F(-, F(-, -)).$$

of the curried functors $(\mathcal{C}^2)^{\text{op}} \times \mathcal{C} \to \mathcal{C}$.

In particular, this applies to the category of simplicial sets, as shown in the lecture (Example II.19 ii). It does not generally apply in topological spaces: A product of quotients is in bijection with the quotient of the product, since $\text{Set}$ is cartesian closed and the forgetful functor $\text{Top} \to \text{Set}$ has both adjoints (namely?), so preserves both limits and colimits, but these spaces may have different topologies! Keep this in mind, when next defining a homotopy between maps out of a quotient space!

**Exercise 2.** Produce a canonical isomorphism of simplicial sets

$$\text{NFun}(\mathcal{C}, \mathcal{D}) \to F(\text{N}\mathcal{C}, \text{N}\mathcal{D})$$

natural in $\mathcal{C}, \mathcal{D} \in \text{Cat}$. Deduce that

1. natural transformations of functors $\eta: F \Rightarrow G$ are in one-to-one correspondence with simplicial homotopies between $\text{N}(F)$ and $\text{N}(G)$,
2. adjoint functors give simplicial homotopy equivalences between their nerves,
3. the map $\Delta^n \to *$ is a homotopy equivalence of simplicial sets, and
4. conjugation by an element $g \in G$ induces a map $c_g: \text{N}\mathcal{B} \to \text{N}\mathcal{B}$ that is homotopic to the identity.

The easiest (if least explicit) way of showing the third item is to note that the degeneracy and face map give adjoint functors $N[n] \leftrightarrow N[m]$ when arranged correctly! This fact is also quite useful in other contexts, and often overlooked.

As group homology can be identified with $H_*(|\text{N}\mathcal{B}|)$, the final item in particular, implies that conjugations act trivially in homology, a standard fact in group theory.
Exercise 3. Let $X$ be a simplicial set with inner horn fillers up to dimension 3. Let furthermore $f, g \in C_1$ be composable 1-simplices, i.e. $d_0g = d_1f$, and $h \in C_1$ with $d_1h = d_1g$ and $d_0h = d_0f$ a potential composite. Show that there exists an $H \in C_2$ with $d_0H = f, d_1H = h$ and $d_2H = g$, if and only if $[f] \circ [g] = [h]$ in $\pi X$.

The 'only if' direction is of course the definition of composition in $\pi C$. You are to show the converse.

Exercise 4. Show that for a topological space $X$ construct an isomorphism $\pi (\text{Sing} X) \rightarrow \pi_1 (X)$ of categories, where the right hand side denotes the fundamental groupoid of $X$, whose objects are the points of $X$ and whose morphisms are homotopy classes of path relative to their endpoints.

If you are unfamiliar with the fundamental groupoid, note that the endomorphisms of an object $x \in \pi_1 (X)$ exactly make up the fundamental group $\pi_1 (X, x)$. 