The first exercise removes the cofibrancy hypothesis from Proposition V.4.17, as Christoph had already announced in the lecture:

**Exercise 1.** Let $I$ be filtered and $\eta: F \Rightarrow G$ a pointwise weak homotopy equivalence of functors $I \to \text{sSet}$. Show that the induced map

$$\text{colim}_I F \to \text{colim}_I G$$

is a weak homotopy equivalence as well. Show also that the analogous statement for pushouts instead of filtered colimits fails without the cofibrancy hypothesis of Exercise 3 on sheet 9.

**Hint:** Use Ex$_\infty$.

**Exercise 2.** Show that a Joyal equivalence $X \to Y$ between simplicial sets induces an equivalence of homotopy categories $hX \to hY$.

The aim of the last two exercises is to prove the classical simplicial approximation theorem and its variant V.5.10.

**Theorem.** For any continuous function $f: |K| \to |L|$ between the realisations of simplicial complexes $K$ and $L$ with $K$ finite, there is an $m \in \mathbb{N}$ and a simplicial approximation $|\varphi|: |K| \cong |\text{bsd}^m(K)| \to |L|$ of $f$.

Here, for simplicial complexes $K$ and $L$ and $f: |K| \to |L|$ a continuous function between their realisations, a simplicial map $|\varphi|: |K| \to |L|$ is a simplicial approximation of $f$, if for each point $x \in |K|$ the image $|\varphi|(x)$ lies in the carrier of $f(x)$ (i.e. in the unique simplex of $L$, s.t. $f(x)$ lies in its interior), and the identification in the last line is given by the homeomorphism $c: \text{bsd}K \cong K$ determined by sending a vertex $v$ of $\text{bsd}K$ to the barycenter of the simplex $v$ of $K$ (and linearly extending).

For the proof we also need the (open) star at a vertex $v$, that is the union of the interiors of all simplices containing $v$, i.e. $\text{star}(v) := \bigcup_{v \in \sigma} \text{int} \sigma$.

**Exercise 3.** Let $f: |K| \to |L|$ be a continuous function between the realisations of simplicial complexes $K$ and $L$, such that the image of
every star in $K$ is contained in a star of $L$. Let furthermore $g: V(K) \to V(L)$ be a map, such that $f(\text{star}(v)) \subseteq \text{star}(g(v))$ for every vertex $v$ of $K$. Show that

- $g$ extends to a simplicial map $K \to L$ and
- that $g$ is a simplicial approximation of $f$.

**Exercise 4.** Assume that a simplicial map $|\varphi|: |K| \to |L|$ is a simplicial approximation of a continuous function $f: |K| \to |L|$. Show that $|\varphi|$ and $f$ are homotopic as maps from $|K|$ to $|L|$ and use Lebesgue’s lemma to deduce the simplicial approximation theorem.

Show furthermore, that $c$ and the last vertex map $\lambda: |\text{bsd}(K)| \to |K|$ are homotopic, and deduce the absolute case of Proposition V.5.10 (i.e. $U = \emptyset$).

Let us spare you the relative cases; it consists of a careful analysis of the constructions involved.