Definition. A morphism \( f : x \to y \) is called an epimorphism or monomorphism, if for all \( g, g' : y \to z \) or \( h, h' : w \to x \) we have
\[
g \circ f = g' \circ f \implies g = g' \quad \text{or} \quad f \circ h = f \circ h' \implies h = h',
\]
respectively.

Exercise 1. Let \( w, x, y, z \in \mathcal{C} \) be four objects and
\[
\begin{array}{ccc}
  w & \xrightarrow{f} & x \\
  \downarrow & & \downarrow \\
  y & \xrightarrow{g} & z
\end{array}
\]
three morphisms.

i) Show that \( f \) is an isomorphism if it is both an epimorphism and monomorphism and in addition admits a retraction or section.

ii) Deduce that if \( g f \) and \( h g \) are isomorphisms then all of \( f, g, h \) and \( h g f \) are isomorphisms as well.

iii) Deduce that an isomorphism admits a unique section, retraction and inverse, all of which agree.

iv) Show by way of an example that a map which is simultaneously an epi- and monomorphism need not be an isomorphism.

Note that part ii), often called the 2-out-of-6-property contains the statement that isomorphisms are closed under composition, since identities are clearly isomorphisms. We used this in the claim that the groupoid core of a (locally small) category is again a category.

Exercise 2. Construct a bijection
\[
\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G) \to \text{Nat}(F, G)
\]
for any two functors \( F, G : \mathcal{C} \to \mathcal{D} \). What does the composition in \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) correspond to under this translation?

Definition. Given a functor \( F : \mathcal{C} \to \mathcal{D} \), let \( \mathcal{S}_F \) and \( \mathcal{R}_F \) be the categories of sections and retractions, respectively: The objects of \( \mathcal{S}_F \) are pairs of a functor \( S : \mathcal{D} \to \mathcal{C} \) and a natural isomorphism \( \tau : F \circ S \Rightarrow \text{id}_\mathcal{D} \), morphisms \( (S, \tau) \to (S', \tau') \) are given by natural transformations \( \sigma : S \Rightarrow S' \), such that the composite
\[
F \circ S \xrightarrow{F \sigma} F \circ S' \xrightarrow{\tau'} \text{id}_\mathcal{D}
\]
equals \( \tau \) and composition is given by composition. \( \mathcal{R}_F \) is defined dually.
Exercise 3. Show that

i) $S_F$ is a groupoid if $F$ is an embedding,

ii) $R_F$ is a groupoid if $F$ is essentially surjective, and

iii) $F$ is an equivalence if and only if both $S_F$ and $R_F$ are equivalent to the trivial category.

Note the similarities to exercise 1.

Definition. For a small category $C$ define the set of components $\pi_0(C)$, to be $\text{obj}(C)/\sim$ where $\sim$ is generated by

$$x \sim y \quad \text{if} \quad \text{Hom}_C(x, y) \neq \emptyset.$$ 

$C$ is called connected if $\pi_0(C)$ has only one element.

Exercise 4. Show that the $2$-essential image, that is the categories that are equivalent to those in the literal image (but not necessarily isomorphic; those form the essential image),

i) of $\mathbb{B}: \text{Mon} \to \text{Cat}$ consists precisely of all categories with connected groupoid core,

ii) of $\mathbb{N}: \text{Ord} \to \text{Cat}$ consists precisely of those categories $C$ for which the combined source-and-target map

$$(s, t): \text{mor}(C) \to \text{obj}(C)^2$$

is injective,

iii) and of $\mathbb{D}: \text{Set} \to \text{Cat}$ consists precisely of the groupoids with injective source-and-target map.

Show by way of an example that the image of a functor need not be a sub-category.

So in a somewhat perverse sense, a set is exactly a partially ordered groupoid.