EXERCISE SHEET NO 8 - ALGEBRAIC TOPOLOGY II

FABIAN HEBESTREIT, THOMAS NIKOLAUS

Exercise 1. Verify the second claim of proposition 7.7 in the lecture, namely that

$$Z^n(C,F)\times_1 B^n(C',F')+B^n(C,F)\times_1 Z^n(C',F') \subseteq B^n(C\otimes C', F\otimes F').$$

Given a map $f : S^{2n-1} \to S^n$, consider the complex

$$\text{hocofib}_f = S^n \cup_f D^{2n},$$

its *homotopy cofibre.* Clearly, $c^* : H^{2n}(S^{2n}) \to H^{2n}(C_f)$, where c collapses the n-cell to a point, is an isomorphism, as is $i^* : H^n(C_f) \to H^n(S^n)$. We define the *Hopf-invariant* $H(f)$ by

$$(i^*)^{-1}(\iota_n)^2 = H(f)c^*(\iota_{2n})$$

By anti-commutativity of the cup-product H vanishes whenever n is odd.

Exercise 2. Show the following:

(1) H forms a homomorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$.
(2) For $g : S^{2n} \to S^{2n}$, we have $H(g \circ f) = \deg(g)^2 H(f)$.
(3) For $h : S^{4n} \to S^{4n}$, we have $H(f \circ h) = \deg(h)H(f)$.
(4) The *Hopf maps*, $\eta : S^3 \to S^2, \sigma : S^7 \to S^4$ and $\nu : S^{15} \to S^8$ all have Hopf-invariant 1.
(5) For $f : S^{4n-2} \to S^{2n-1}$ we have $H(Sf) = 0$.

Recall that the Hopf fibrations in general are the projection maps $\pi : S^{(n+1)d-1} \to \mathbb{K}P^n$, sending a vector to its associated line, where $\mathbb{K}$ is a d-dimensional division algebra (i.e. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ (in which case $n = 1$!). The Hopf maps above arise from this by the usual identification $\mathbb{K}P^1 = S^d$. In case you have not heard about the octonions and their projective space(s), just do the others (but get to know them, for example via the excellent blog post at

http://math.ucr.edu/home/baez/octonions/octonions.html

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which also explains the importance of the Hopf maps in great detail.

We shall now see two ways to produce elements of even Hopf-invariant in any dimension. Given the standard cell decomposition of S^n with one 0- and one n-cell, we find the product cell structure on $S^n \times S^m$ with one 0-cell, an n and am m-cell, and finally one $n+m$-cell; in particular $(S^n \times S^m)^{(n+m-1)} = S^n \vee S^m$. We therefore have an attaching map $S^{n+m-1} \to S^n \vee S^m$, sometimes called a *Whitehead map*. We can use it to produce a product on homotopy groups: Given $f \in \pi_n(X)$ and $g \in \pi_m(X)$ we can form

$$[f, g] \in \pi_{n+m-1}(X), S^{n+m-1} \to S^n \vee S^m \to X \vee X \to X$$

and a similar exterior version $[-,-] : \pi_n(X) \otimes \pi_m(Y) \to \pi_{n-m-1}(X \vee Y)$.

Exercise 3. Show the following:

(1) This makes $\pi_{*+1}(X)_{ab}$ into a graded Lie-algebra.
(2) $H([\mathrm{id}_{S^{2n}}, \mathrm{id}_{S^{2n}}]) = 2$
(3) Determine the Lie-Algebra structure of $\pi_{*+1}(S^n) \otimes \mathbb{Q}$.

In particular, the rank of $\pi_{4n-1}(S^{2n})$ is at least 1.

Let $\mathcal{G}(n)$ denote the monoid of pointed self homotopy equivalences $S^n \to S^n$. One-point-compactification clearly gives a map $J : O(n) \to \mathcal{G}(n)$, Whitehead's *J-homomorphism*.

Exercise 4. Show the following:

(1) Adjoining coordinates gives a natural isomorphism

$$\mathrm{adj} : \pi_k(\mathcal{G}_n) \to \pi_{k+n}(S^n),$$

whenever $k > 0$.

(2) Given an element $f \in \pi_{2n-1}(\mathrm{SO}(2n))$ we have

$$H(\mathrm{adj}(J(f))) = e(V_f),$$

where V_f denotes the vector bundle over S^{2n} obtained from f via the *clutching construction* and e its *Euler number*.

Since a vector bundle uniquely determines the homotopy class of its clutching function one also writes $J(E)$ instead of $J(f)$ if $V_f \cong E$.

(3) In particular, $H(J(TS^{2n})) = 2$.

Hint: Show that the homotopy cofibre of $\mathrm{adj}(J(f))$ is the Thom-space of V_f and recall that one definition of the Euler class $e(E) \in H^d(B)$ of a d-dimensional vector bundle is that $th(e(E)) = u(E)^2$, where

$u(E) \in H^d(Th(E), *)$ denotes the *Thom class* of E and th the Thom isomorphism $H^*(B) \to H^{*+d}(Th(E), *)$.

Here are some closing remarks, that are slightly beyond the scope of this exercise sheet:

(1) The two elements $[\mathrm{id}_{S^n}, \mathrm{id}_{S^n}]$ and $\mathrm{adj}(J(TS^n))$ in $\pi_{2n-1}(S^n)$ actually agree (can you show that?) and are usually called w_n.
(2) An element $x \in \pi_{2n-1}(S^n)$ is a suspension precisely if $H(x) = 0$.
(3) Suspensions of Whitehead products always vanish.
(4) The kernel of the surjection
$$S : \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1}) = \mathbb{S}_{n-1}$$
is generated by the Whitehead square w_n at least up to odd torsion.
(5) The order of $w_{2n-1} \in \pi_{4n-3}(S^{2n-1})$ for odd n agrees with the index of the image of $H : \pi_{4m-1}(S^{2n}) \to \mathbb{Z}$, thus generating another interesting element in the homotopy groups of spheres, whenever there is no element of Hopf-invariant one.

Except for the first, all of these statements are simple consequences of the *EHP-(spectral)-sequence*, a powerful tool in the computation of unstable homotopy groups, that we will not have time to cover.