EXERCISE SHEET NO 5 - ALGEBRAIC TOPOLOGY II

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Exercise 1. Compute $H_*(\mathbb{R}P^\infty, \mathcal{Z})$ and $H^*(\mathbb{R}P^\infty, \mathcal{Z})$ where $\mathcal{Z}$ is the local coefficient system given by the fibration

$$S^\infty \times_{\mathbb{Z}/2} \mathbb{Z} \to \mathbb{R}P^\infty.$$

Exercise 2. Give a counterexample to the claim of Serre's rational Hurewicz theorem, when the space X in question is not required to be simply connected, but only to have abelian, torsion fundamental group.

Exercise 3. Let $A = \mathbb{F}_2[x, y_i | i \in \mathbb{N}]$ where $|y_i| = 2^i - 1$ and $|x| = 1$. Make it into a differential, graded algebra by setting

$$d(x) = y_0, d(y_0) = 0, d(y_{i+1}) = y_i^2$$

and extending by the Leibniz rule. Compute $H_*(A, d)$ as an algebra by choosing a clever filtration and using the spectral sequence of a filtered complex.

Exercise 4. Show, via an analysis of the Postnikov tower of S^3, that for any prime number p we have $\pi_k(S^3)_p = 0$ for $k < 2p$ and more importantly that $\pi_{2p}(S^3)_p$ is cyclic of order p. In particular, S^2 and S^3 have infinitely many non-vanishing homotopy groups!

Hint: As a warm-up show that $\pi_4(S^3) = \mathbb{Z}/2$.

Due date: May 29th