EXERCISE SHEET NO 5 - ALGEBRAIC TOPOLOGY II

FABIAN HEBESTREIT, THOMAS NIKOLAUS

Exercise 1. Which of the following are Serre-classes and which of those are good?

(1) torsion groups
(2) torsionfree groups
(3) finite groups
(4) p-groups
(5) finitely generated groups
(6) divisible groups

We want to characterise Serre-classes in a more categorical flavour:

Definition. A functor F between abelian categories is called *exact* if it sends exact sequences to exact sequences. Its *kernel* is the full subcategory of the source spanned by the objects X with $F(X)$ a zero object.

Exercise 2. Show that Serre classes are precisely the kernels of exact functors.

Hint: One direction is easy, the other a bit more difficult (and only true if one allows *large* categories, i.e. does not require the morphisms between two objects to form a set!). Given a Serre-class $\mathcal{D} \subseteq \mathcal{A}$ one can proceed as follows to produce an projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{D}$ (i.e. defining the right hand category!):

The class

$$\mathcal{S}_{\mathcal{D}} = \{f \in \mathcal{A} \mid \ker(f), \operatorname{coker}(f) \in \mathcal{D}\}$$

is a *multiplicative system*, that is a subcategory of $\mathcal{A}$ (containing all objects) such that i) every solid (dotted) diagram

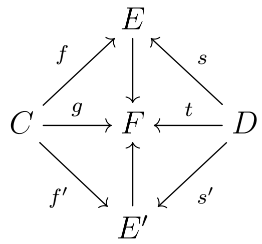
$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \vdots \\ E & \dashrightarrow & F \end{array}$$

with left (resp. right) hand vertical arrow in $\mathcal{D}$ may be extended as indicated with right (resp. left) hand vertical arrow in $\mathcal{D}$ as well and

Due date: June 12th

ii) for every pair of morphisms $f, g\colon C \to D \in \mathcal{A}$, there exists an $s\colon E \to C \in \mathcal{D}$ with $fs = gs$ if and only if there exists a $t\colon D \to F \in \mathcal{D}$ with $tf = tg$.

For any multiplicative system (the part outside the parathesis already suffices) $\mathcal{S} \subseteq \mathcal{A}$ one can form a category $\mathcal{S}^{-1}\mathcal{A}$ with the same objects as $\mathcal{A}$ and morphisms $C \to D$ given by pairs $s\colon D \to E \in \mathcal{S}$ and $f\colon C \to E$ under the equivalence relation $(s, f) \sim (s', f')$ if there exists (t, g) and a diagram



Now $\mathcal{A}/\mathcal{D} = \mathcal{S}_{\mathcal{D}}^{-1}\mathcal{A}$ does the job.

Also: Do not verify all of the above claims. Pick the steps you think are crucial!

Recall that for a group G and a $\mathbb{Z}G$-module M one has the *group homology* and *cohomology*

$$H_*(G; M) = \mathrm{Tor}^*_{\mathbb{Z}G}(\mathbb{Z}, M) \quad \text{and} \quad H^*(G; M) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M),$$

the derived functors of taking coinvariants and invariants.

Exercise 3. Clearly a $\mathbb{Z}G$-module M determines a local system on BG, that we will also call M. Show that that $H_*(G; M) = H_*(BG; M)$ and $H^*(G; M) = H^*(BG; M)$. Here the left hand side denotes group (co)homology and the right hand side (co)homology with local coefficients.

Exercise 4. Use the homological Serre spectral sequence to compute $H_*(\Omega S^n)$ for all n.