EXERCISE SHEET NO 5 - ALGEBRAIC TOPOLOGY II

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Exercise 1. Use the spectral sequence of a double complex to show that $\mathrm{Tor}_R(M, N)$ is independent of which variable is used to resolve it.

The same argument can essentially be used for any bifunctor, in particular Ext, but let's not go there.

Exercise 2. Given a map $f : X \to Y$, construct a functor

$$\Pi_1(Y) \to \mathrm{Ho(Top)}$$

that sends a point $p \in Y$ to $\mathrm{hofib}_p(f)$, as explained in the lecture.

Definition. A *divided power structure* on an ideal I of an algebra A is a collection of operations $\gamma_i : I \to A$ with the following properties:

(1) $\gamma_0 = 1, \gamma_1 = \mathrm{id}, \gamma_n(I) \subseteq I$ for $n \geq 2$
(2) $\gamma_n(x + y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y)$
(3) $\gamma_n(ax) = a^n\gamma_n(x)$ for $a \in A$
(4) $\gamma_m(x) \cdot \gamma_n(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$
(5) $\gamma_n(\gamma_m(x)) = \frac{(mn)!}{n!(m!)^n}\gamma_{nm}(x)$

These are very important in the theory of infinitesimal deformations in algebraic geometry. We introduce them here, since the rings underlying divided power algebras also frequently show up in the (co)homology of interesting spaces (not always for clear reasons!).

Exercise 3. Show the following:

(1) $n!\gamma_n(x) = x^n$ (hence the name).
(2) Every ideal in a $\mathbb{Q}$-algebra admits a unique structure of a divided power algebra.
(3) The forgetful functor from divided-power-algebras to algebras with a chosen ideal (that is preserved by morphisms!) admits a left adjoint Γ, the *divided power envelope*.
(4) $\Gamma(\mathbb{Z}[x], (x)) = \mathbb{Z}\left[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots\right]$.
(5) $\Gamma(\mathbb{F}_2[x], (x)) = \Lambda_{\mathbb{F}_2}[x, \gamma_2(x), \gamma_4(x), \gamma_8(x), \dots]$.

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We will from now on write

$$\Gamma_R(x_1, ..., x_n) := \Gamma(R[x_1, ..., x_n], (x_1, ..., x_n))$$

and refer to it as the *free divided power algebra* on generators x_i over R (and similar for a countable number of generators). It is readily checked that it represents the functor sending a divided power algebra (R, I, γ) to I^n and thus deserves the name.

Exercise 4. Use the Serre spectral sequence to show that

$$H^*(\Omega^2 S^3, \mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(x_1, x_3, x_7, x_{15}, \dots),$$

i.e. an exterior algebra on one generator in each degree of the form $2^i(2^k - 1)$.

Hints: The divided power algebra is really only a tool for grouping the generators together in this case; if you find a way around it, by all means use it. Recall that $H^*(\Omega S^3, \mathbb{Z}) = \Gamma_{\mathbb{Z}}(y_2)$. A moment's thought will then reveal that $H^*(\Omega S^3, \mathbb{F}_2) = \Gamma_{\mathbb{F}_2}(y_2)$, i.e an exterior algebra with one generator in each degree $2, 4, 8, 16, \dots$.

As a little helper for the calculation: The differentials turn out to be

$$d_{2^k}(\gamma_{2^l}(x_{2^k-1})) = \gamma_{2^{k-1}}(y_2) \left(\prod_{h<l} \gamma_{2^h}(x_{2^k-1}) \right),$$

though this is easier seen in an example than parsed.

Also: You may treat this exercise as a bonus, rather than mandatory.