EXERCISE SHEET NO 3 - ALGEBRAIC TOPOLOGY II

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Exercise 1. Compute $H^*(K(\mathbb{Z},3),\mathbb{Z})$ including its ring structure for as many successive degrees as you can using the Serre spectral sequence for the fibre sequence

$$K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3).$$

Exercise 2. Let X denote the homotopy fibre of the map $S^3 \to K(\mathbb{Z},3)$, that classifies the $[S^3] \in H^3(S^3,\mathbb{Z})$. Compute the ring $H^*(X,\mathbb{F}_p)$ for all p using the Serre spectral sequence for the fibre sequence

$$K(\mathbb{Z},2) \to X \to S^3.$$

As a bonus: What can you now deduce about $H^*(K(\mathbb{Z},3),\mathbb{Z})$ from the Serre spectral sequence of the fibre sequence

$$X \longrightarrow S^3 \longrightarrow K(\mathbb{Z},3)?$$

Recall that we set

$$Z_k^n = \{x \in E_k \mid d_i([x]_i) = 0 \ \forall k \leq i \leq n\}$$
$$B_k^n = \{x \in Z_k^n \mid [x]_n \in \mathrm{im}(d_n)\}$$

and

$$Z_k^\infty = \bigcap_{n\geq k} Z_k^n \text{ and } B_k^\infty = \bigcup_{n\geq k} B_k^n$$

for a general spectral sequence (E,d).

Exercise 3. Show one of the following:

(1) $Z_k^n/B_k^n \cong E_{n+1}$ for all $n \geq k$ via the canonical projections.
(2) $Z_k^\infty/B_k^\infty \cong Z_l^\infty/B_l^\infty$ for all $l \geq k$ via the canonical projections.

Given a cochain complex (C,d) of $\mathbb{Z}$-graded groups with the property that

$$d^n(C^{n,k}) \subseteq \bigoplus_{i\geq k} C^{n+1,i},$$

consider the associated filtered chain complex with

$$F_k^* = \bigoplus_{i\geq k} C^{*,k};$$

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this is a (canonically split) descending filtration by subcomplexes since $d^n(F^n_k) \subseteq F^{n+1}_k$ translates precisely into the condition above. Represent the differential $d^n : C^n \to C^{n+1}$ by an infinite block matrix D^n, i.e. $D^n_{i,j}$ consists of the homomorphism $C^{n,i} \to C^{n+1,j}$ given by

$$C^{n,i} \xrightarrow{\iota} \bigoplus_k C^{n,k} \xrightarrow{d^n} \bigoplus_k C^{n+1,k} \xrightarrow{\pi} C^{n+1,j}$$

and recall that it is lower triangular.

Exercise 4. Verify the claim from the lecture, that in the spectral sequence $\mathcal{L}(C, F)$ the ith differential is induced by the $i-1$st off-diagonal in the matrices D^n.