EXERCISE SHEET NO 10 - ALGEBRAIC TOPOLOGY II

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Exercise 1. Show that $\beta_2 = \mathrm{Sq}^1$.

Of course you should not simply appeal to theorem 8.3 of the lecture. However, none of the calculations culminating in the determination of $H^*(K(\mathbb{Z}/2, n))$ made use of the identification in question, so may be used, if you do not want to appeal to the constructions.

Exercise 2. Show that for any two abelian groups the following diagram of isomorphisms commutes:

$$\begin{array}{ccc} \Xi(A,B) & \xrightarrow{\theta} & \mathrm{Ext}(A,B) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle h} \\ \mathcal{O}(A,n,B,n+1) & \xrightarrow[e]{} & H^{n+1}(K(A,n),B) \end{array}$$

Here $\Xi(A,B)$ denotes the set of isomorphism classes of extensions of A by B, $\mathcal{O}(A,n,B,n+1)$ the set of cohomology operations

$$H^n(-,A) \to H^{n+1}(-,B),$$

θ assigns to an extension $\zeta : B \to E \to A$ the element $\delta_\zeta(\mathrm{id}_A)$, where

$$\delta_\zeta \colon \mathrm{Hom}(A,A) \to \mathrm{Ext}(A,B)$$

is the boundary operator of the long exact sequence associated to ζ, h is the map featuring in the universal coefficient theorem, β assigns to an extension its Bockstein homomorphism and e is given by evaluation on the fundamental class $\iota_n \in H^n(K(A,n),A)$.

Exercise 3. Use the Steenrod operations to show that $S^3 \times \mathbb{C}P^\infty$ and $S^1 \times \mathbb{C}P^\infty / S^1 \times *$ are not homotopy equivalent.

Exercise 4. Show that the suspension homomorphism $\pi_5(S^2) \to \pi_6(S^3)$ is injective, by considering the homotopy fibre of the suspension map $S^2 \to \Omega S^3$.

Due date: July 10th