EXERCISE SHEET NO 1 - ALGEBRAIC TOPOLOGY II

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We recall the notion of a fibration:

Definition. Let $f : E \longrightarrow B$ be a continuous function and Z some topological space. f is said to admit *homotopy lifting* w.r.t. Z, if every commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & E \\ \downarrow{\scriptstyle i_0} & & \downarrow \\ Z \times I & \longrightarrow & B \end{array}$$

admits another continuous function $Z \times I \to E$ making both arising triangles commute as well. A *fibration* (sometimes called a *Hurewicz fibration* to set it apart from the Serre fibrations to be defined momentarily) is a map f that admits homotopy lifting for every space Z. f is called a *Serre fibration*, if it admits homotopy lifting for all cubes I^n, $n \geq 0$.

It is readily checked, that Serre fibration admit homotopy lifting for all cell complexes. Dual to the notions of homotopy lifting and fibration there are the notions of homotopy extension and cofibration, which will, however, not play a large role in the present course.

Exercise 1. Let $f : E \to B$ be a Serre fibration and $B' \subseteq B$. Show that the projection

$$f_* : \pi_n(E, f^{-1}(B'), e) \to \pi_n(B, B', f(e))$$

is an isomorphism for every base point $e \in E$ with $f(e) \in B'$. In particular, one obtains a long exact sequence

$$\cdots \to \pi_n(F, e) \to \pi_n(E, e) \to \pi_n(B, f(e)) \to \pi_{n-1}(F, e) \to \dots$$

associated to any Serre fibration.

Maps $f : E \to B$ with the property that

$$f_* : \pi_n(E, f^{-1}(f(e)), e) \longrightarrow \pi_n(B, f(e))$$

is an isomorphism for all $e \in E$ are called *quasifibrations.* Aside from Serre fibrations there is one more prominent class of examples due to

Due date: May 1st

a theorem of Dold and Thom. We shall refrain from discussing it, however, let us mention that there is an entire zoo of such notions: There are homotopy fibrations, micro fibrations, weak quasifibrations, pointed fibrations, local fibrations, local homology fibrations and the list goes on. We will probably encounter none of them.

Exercise 2. Show that every fibre sequence with path connected base space admits a long exact sequence as above as well and conclude, that for every $b \in B$ the sequence $f^{-1}(b) \to E \xrightarrow{f} B$ together with the constant null homotopy is a fibre sequence, whenever f is a quasifibration.

The order of the two items to be shown can also be changed, for example by the process known as *fibrant replacement.* This is a perfectly acceptable solution, as well.

Exercise 3. Given a map $f : E \to B$ and an open cover $\mathcal{U}$ of B such that $f_{|f^{-1}(U)}$ is a Serre fibration for every $U \in \mathcal{U}$ show that f is Serre fibration as well. Conclude: Fibre bundles are Serre fibrations.

Hint: Subdivide the cube cleverly using the Lebesque covering lemma.

The appropriate version for Hurewicz fibrations requires the cover to be numerable, i.e. to admit a subordinate partition of unity, but the proof is more work. It is *not* true, that arbitrary fibre bundles are Hurewicz fibrations. Clearly, these subtleties disappear when the base is a paracompact space.

Exercise 4. Let $p : S^2 \to K(\mathbb{Z}, 2)$ classify a generator of $H^2(S^2, \mathbb{Z})$. Show that its homotopy fibre is simply connected and use the Serre spectral sequence to compute the cohomology of its homotopy fibre. Can you guess and prove, what that fibre is?