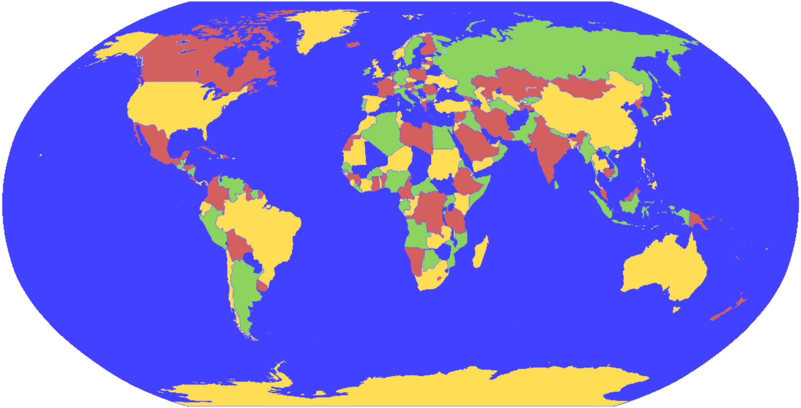
Three colors suffice

Or: SO3 webs in action

~~Accept~~Change what you cannot ~~change~~accept

I report on work of Tait, Temperley–Lieb, Yamada & Turaev, and many more

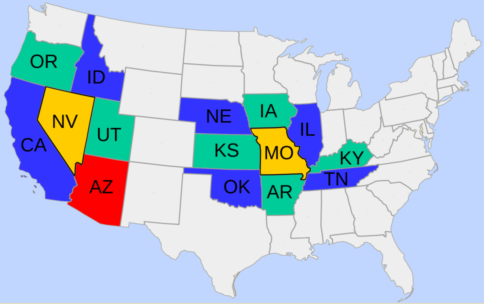
Tait's webs

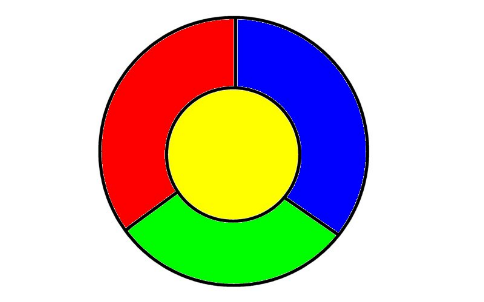


- Task Color countries such that two countries that share a border get different colors
- Above A four coloring of the world (counting the ocean as a country)
- Question How many colors are needed when varying over all maps?

Tait's webs

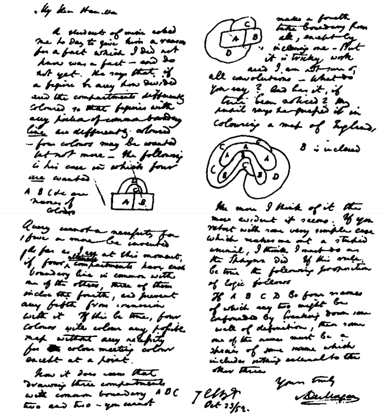
It is easy to see that one needs at least four colors:





- Task Color … et different colors
- Above A f … a country)
- Question I … maps?

Tait's webs



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented — As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary ABC two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

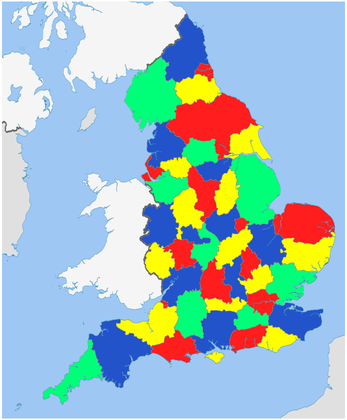
B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly
A De Morgan

7 CBT
Oct 23/52.

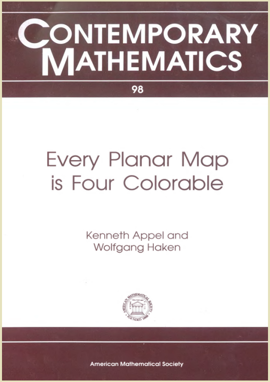


- Guthrie ∼1852 was coloring counties of England
- They conjectured that only four colors are needed and wrote De Morgan
- De Morgan popularized the question

Tait's webs

The 4CT ("four colors suffice") was the first major theorem with a computer assisted proof

Appel–Haken's proof ∼1976 has
770ish pages
with 500 pages of :
"cases to check"
The book is from ∼**1989**



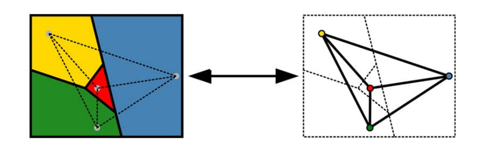
Today A different proof due to Tait & Temperley–Lieb & Yamada & Turaev

Spoiler The proof also has a computer component

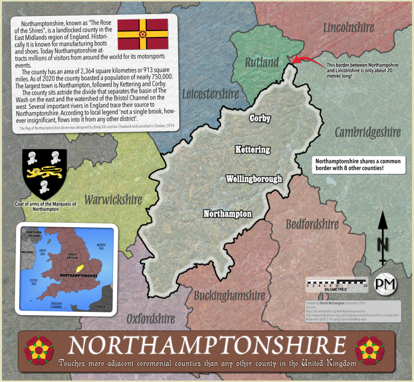
- De Morgan popularized the question

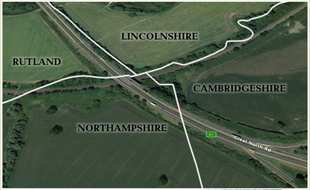
Tait's w

How to go to graphs? You know the drill:



Here is an example why this is actually cool:





Northamptonshire borders Lincolnshire with a ≈20m border

That is impossible to see on the real map!

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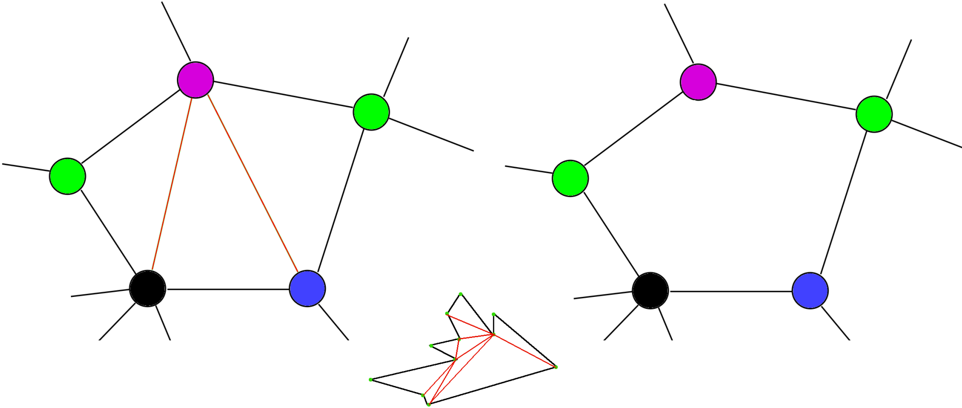
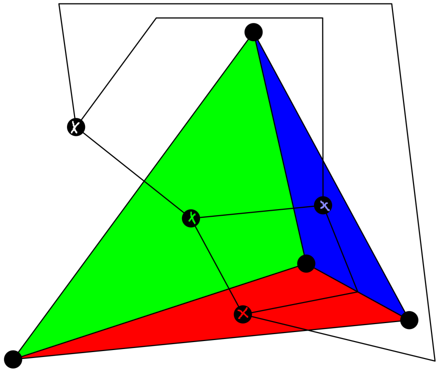
Tait's webs



- Math formulation Every planar graph is four vertex-colorable
- **Tait ∼1880** We can restrict to triangulated planar graphs
- Why? We can keep the coloring after removing edges!

Tait's webs



- **Tait** $\sim$**1880** We have

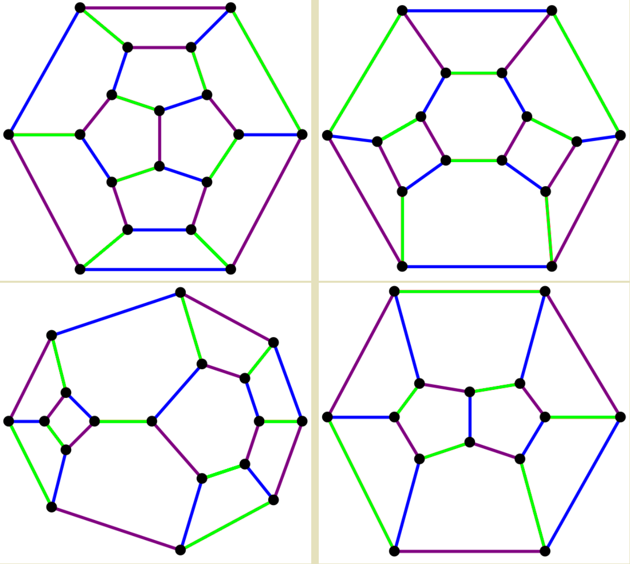
4CT (vertices) $\Leftrightarrow$ triangulated 4CT (vertices) $\Leftrightarrow$ trivalent 4CT (faces)

- Why? The dual of a triangulated planar graph is a trivalent planar graph

Tait's webs

Tait's webs = trivalent planar graphs (the name came later)

Examples



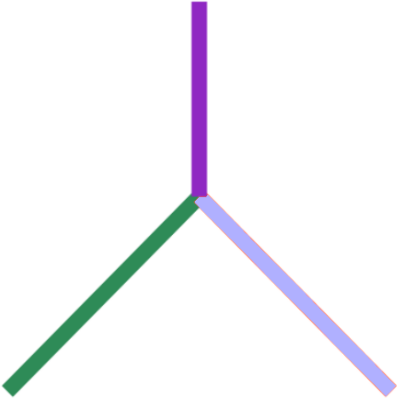
▶ Tait ∼

4CT … T (faces)

▶ Why? … ar graph

Tait's webs

3 edge coloring :

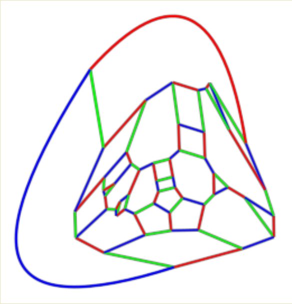


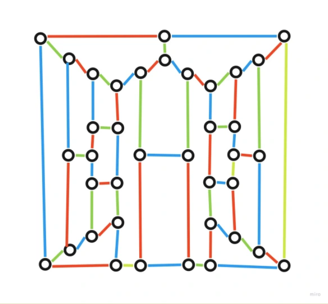
- 3CT is the statement that every trivalent planar bridgeless graph admits a 3 edge coloring
- **Tait $\sim$1880** We have

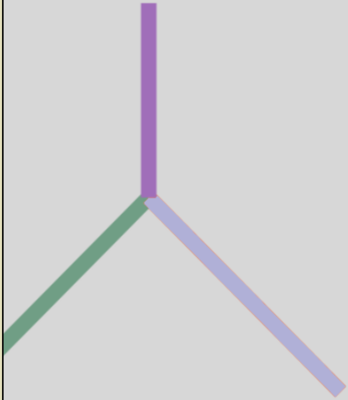
$$\text{4CT (vertices)} \Leftrightarrow \text{3CT (edges)}$$

Tait’s

Examples







- ery trivalent planar bridgeless graph admits a 3
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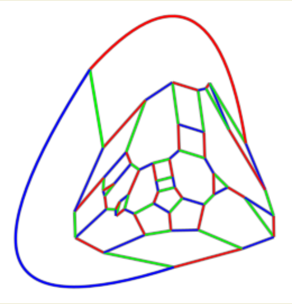
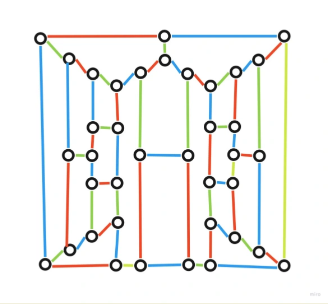
ertices) ⇔ 3CT (edges)

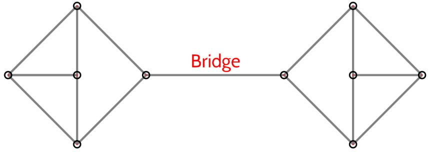
Tait's

Examples





Bridge = edge that disconnects the graph after removal



"Bridges $\leftrightsquigarrow$ countries touching themselves" so we disregard them

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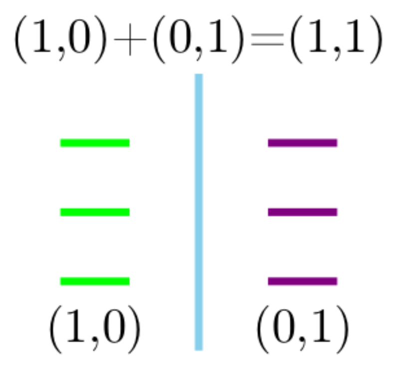
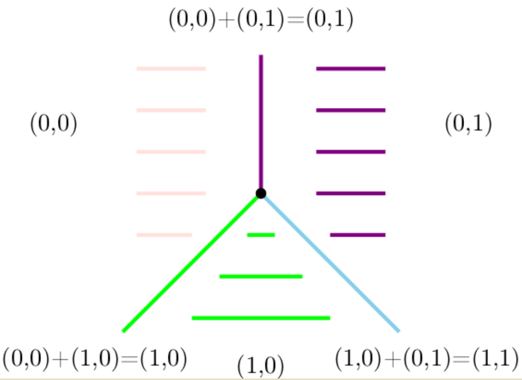
ertices) ⇔ 3CT (edges)

Proof sketch: 4CT ⇒ 3CT

Identify the 4 colors with elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
and use the rule $(a) + (b) = (a + b)$:



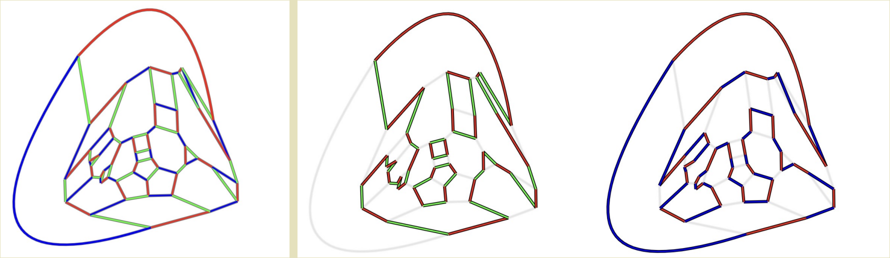


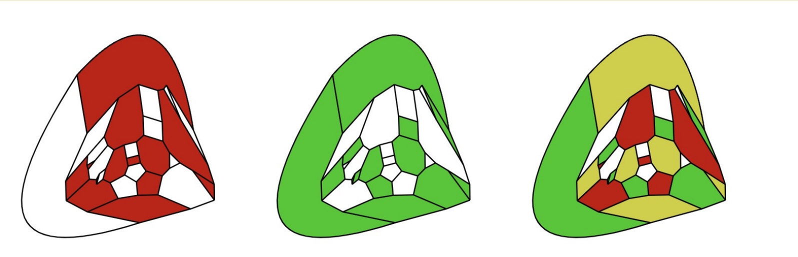
$+$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

.

Proof sketch: 4CT $\Leftarrow$ 3CT

Identify the 4 colors with elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
get "two-color subgraphs" color and overlay them
and use the rule $(a) + (b) = (a + b)$:

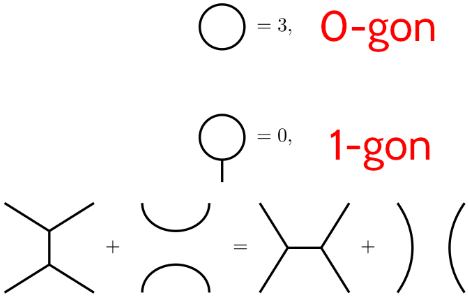




Temperley–Lieb's webs

gen :

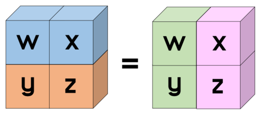
rels :

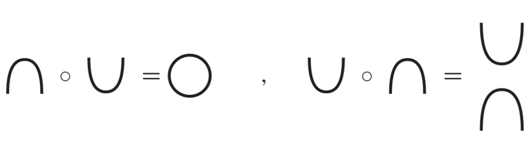


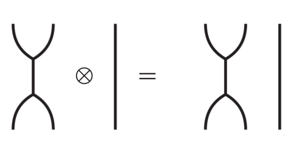
- Define a category **Web**$(\mathrm{SO3})$ with: Objects $= \bullet^{\otimes n}$; Morphisms $=$ everything you can get from a trivalent vertex
- Make it $\mathbb{Z}$-linear and quotient by the relations above + isotopies

Te

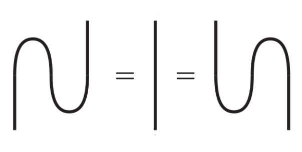
We define this as a monoidal category with
$\circ$ = vertical stacking, $\otimes$ = horizontal juxtaposition

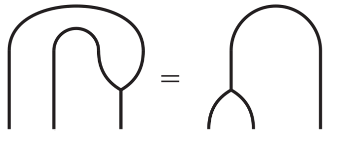






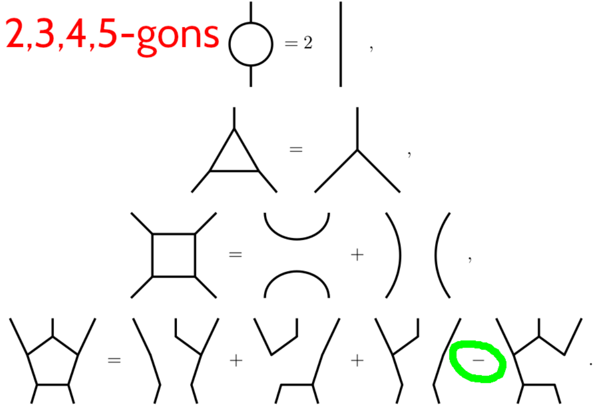
Isotopy relations are of the form





and the interesting relations are the combinatorial ones

Temperley–Lieb's webs



- The above relations hold
- Upshot Every web evaluates to a number in **Web**$(\mathrm{SO3})$ Web evaluation
- **Essentially done by Temperley–Lieb** $\sim$**1971** The web evaluation counts the number of 3-colorings

Temperley–Lieb's w

2,3,4

Why are 0-5 gon relations enough? Well:

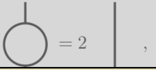
Lemma Every web contains a ≤ 5 gon

Proof Use the Euler characteristic

This is a closed web!

- The above relat
- Upshot Every … Web evaluation
- **Essentially do** … evaluation counts the number of 3-colorings

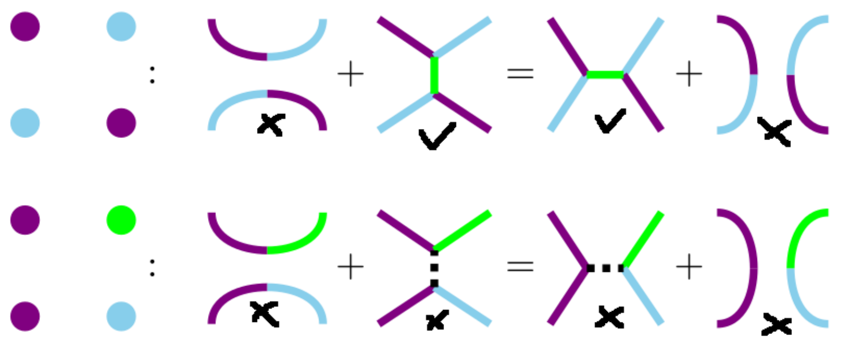
2,3,4,5-gons



Why does the evaluation count colorings? Well:

Lemma The relations preserve the number of colors

Proof Color the boundary and check, e.g.:

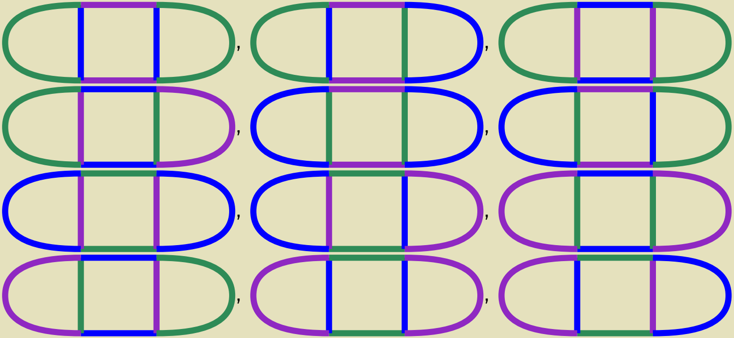


- **Essentially done by Temperley–Lieb ∼1971** The web evaluation counts the number of 3-colorings

Example

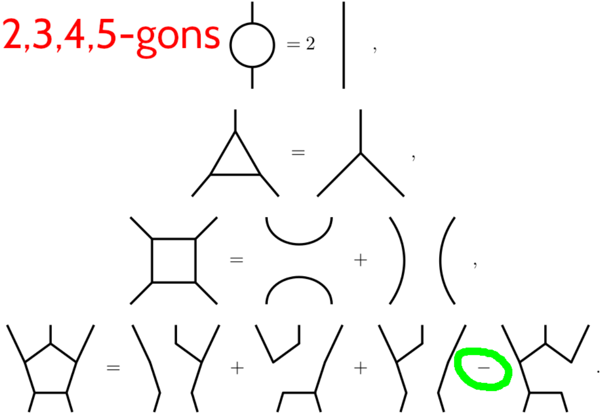
$= 3 + 3^2 = 12$

We have indeed twelve 3-colorings:



- **Essentially done by Temperley–Lieb ∼1971** The web evaluation counts the number of 3-colorings

Temperley–Lieb’s webs

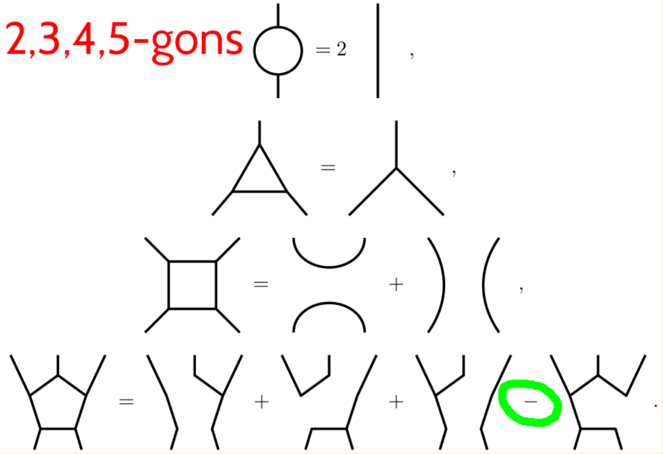


- **Essentially done by Temperley–Lieb ∼1971** 4CT ⇔ every webs evaluates to a nonzero scalar
- The 4CT is then almost immediately true but there is a sign in the pentagon relation, and there might be cancellations

Two questions remain:

Question 1 Can we beef this up into a proof of 4CT?

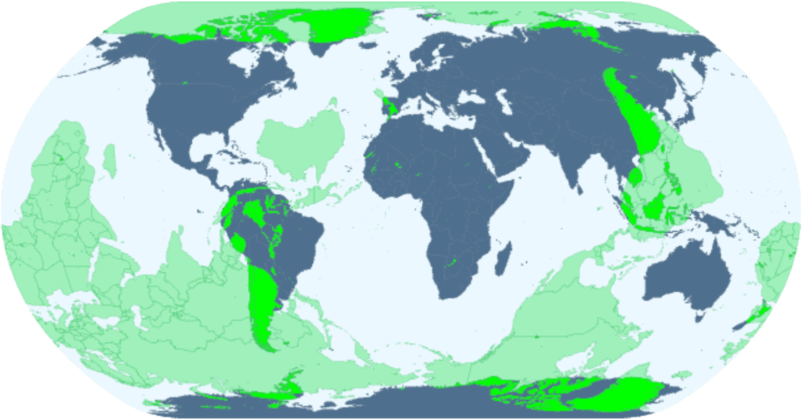
Question 2 Where do these relations come from?



▶ Essen... evaluates to a n...

▶ The 4... pentagon relation, and there might be cancellations

Yamada & Turaev's webs

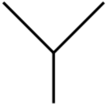


- $\mathrm{SO3}$ = rotations of $\mathbb{C}^3$
- The real version is topologically $\mathrm{SO3} \cong S^2$/antipodal points
- $\mathrm{SO3}$ acts on $\mathtt{X} = \mathbb{C}^3$ by matrices

Yamada & Turaev's webs

$\bullet \mapsto \mathtt{X}$

 $\mapsto$ inclusion of $\mathbb{1}$ into $\mathtt{X} \otimes \mathtt{X}$

 $\mapsto$ inclusion of $\mathtt{X}$ into $\mathtt{X} \otimes \mathtt{X}$

cap =  $: \mathtt{X} \otimes \mathtt{X} \to \mathbb{1}$, cup =  $: \mathbb{1} \to \mathtt{X} \otimes \mathtt{X}$,

tup =  $: \mathtt{X} \otimes \mathtt{X} \otimes \mathtt{X} \to \mathbb{1}$, tdown =  $: \mathbb{1} \to \mathtt{X} \otimes \mathtt{X} \otimes \mathtt{X}$

- **Essentially done by Yamada & Turaev ∼1989** We have **Web**$(\mathrm{SO3}) \cong$ **Rep**$(\mathrm{SO3})$ using the above
- Equivalence of categories $=$ they encode the same information
- **Rep**$(\mathrm{SO3})$ $=$ fd reps of SO3 over $\mathbb{C}$

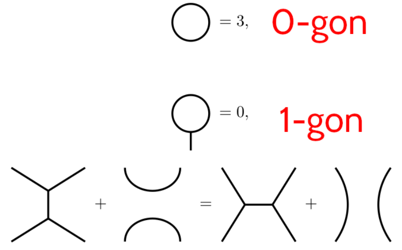
Yamada

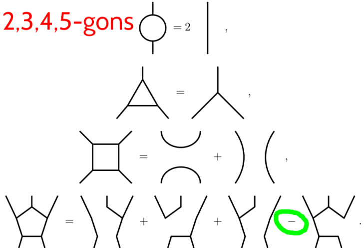
cap = … X,

tup = … $X \otimes X$

- Ess… We…
- Equ…
- Re…

Upshot Under **Web**$(\mathrm{SO3}) \cong$ **Rep**$(\mathrm{SO3})$ the relations, e.g.

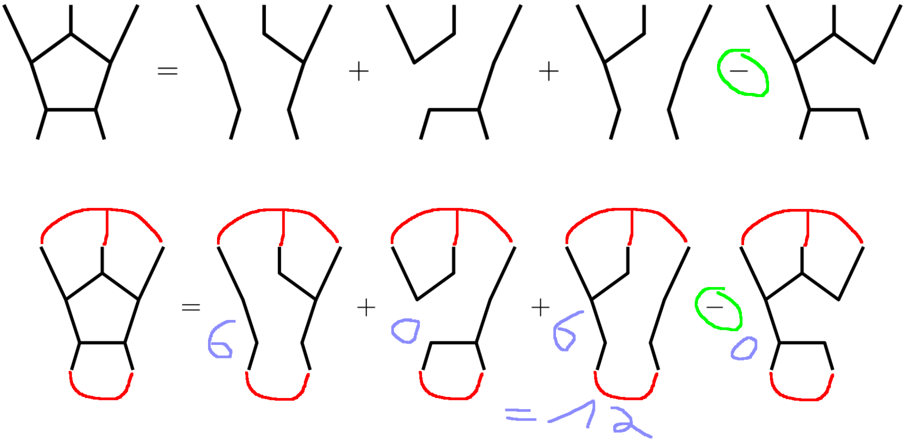




are equations between $\mathrm{SO3}$-equivariant matrices

Example The "I=H" relation is an equation of four 9-by-9 matrices

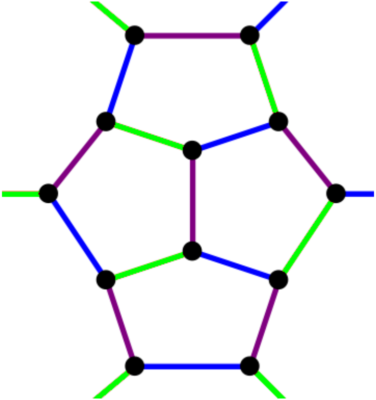
Yamada & Turaev's webs



- Recall that we want to show positivity but the pentagon has a sign
- Observation Closing the pentagon "minimally" evaluates to a positive number

Yamada & Turaev's webs

An unavoidable configuration:



- An Euler characteristic argument shows that there are finitely many "unavoidable configurations" one needs to check similarly
- There are finitely many "minimally closures" for each one of them

Yamada & Turaev's webs

Theorem

(A) 4CT holds ⇔ (B) finitely many configurations evaluate to a positive number

One can computer verify that (B) holds (too many to do by hand)

▶ There are finitely many "minimally closures" for each one of them

Yamada & Turaev's webs

Khovanov–Kuperberg ∼1999 essentially showed
that 3-colorings correspond to a base-change matrix
between a standard basis and a dual canonical basis in **Rep**$(\mathrm{SO3})$

Example

$\{v_1, v_2, v_3\}$ basis of $\mathtt{X}$

Identify:  $= v_1$,  $= v_2$,  $= v_3$

 $\rightsquigarrow v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3 \in \mathtt{X} \otimes \mathtt{X}$

 $\rightsquigarrow$ scalar in front of $v_1 \otimes v_1$

 $\rightsquigarrow$ scalar in front of $v_2 \otimes v_2$

 $\rightsquigarrow$ scalar in front of $v_3 \otimes v_3$

Theorem

(A) 4CT holds $\Leftrightarrow$ (C) finitely many inequalities of matrix entries

One can computer verify that (C) holds (too many to do by hand)

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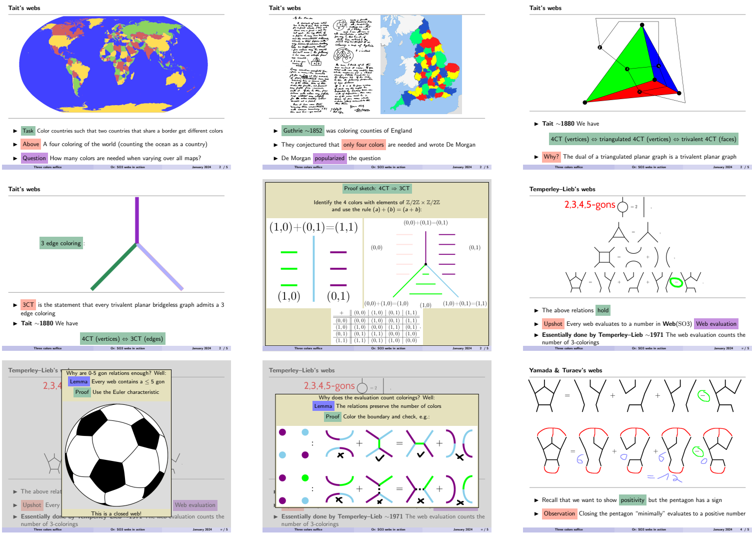
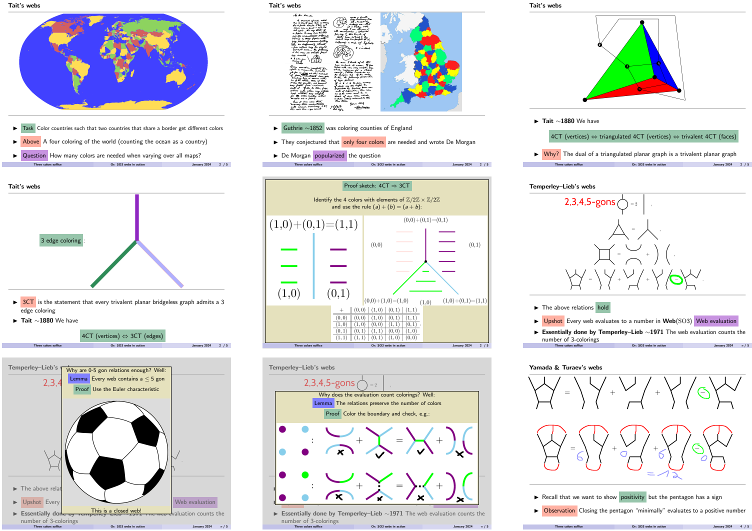
  “una...

- There are finitely many “minimally closures” for each one of them



There is still much to do...



Thanks for your attention!