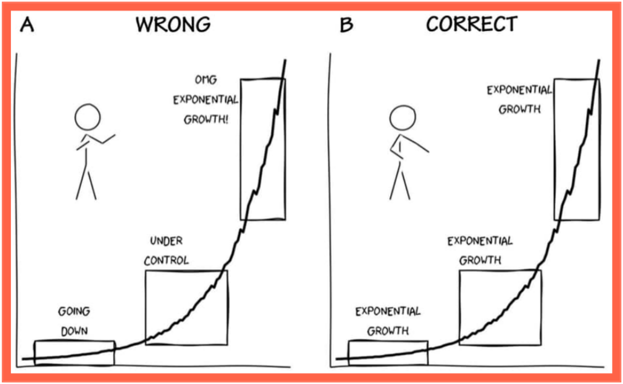
Asymptotics and tensor products

Or: I love matrices

~~Accept~~Change what you cannot ~~change~~accept



I report on work of Kevin Coulembier, Pavel Etingof and Victor Ostrik, and Abel Lacabanne and Pedro Vaz

Let us not count!

- Γ = something that has a tensor product (more details later)
- $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Let us not count!

$\dim_{\mathbb{K}} V = 1$ works perfectly well

but then my story about exponential growth is flawed

so I ignore $\dim_{\mathbb{K}} V = 1$ and assume $\dim_{\mathbb{K}} V > 1$

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Examples of what Γ could be

Any finite group, monoid, semigroup

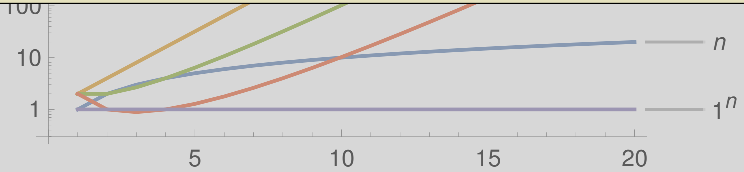
Symmetric groups, alternating groups, cyclic groups, the monster, $GL_N(\mathbb{F}_{p^k})$, ...

Actually any group, monoid, semigroup

$GL_N(\mathbb{C})$, $GL_N(\mathbb{R})$, $GL_N(\overline{\mathbb{F}_{p^k}})$, symplectic, orthogonal, braid groups, Thompson groups, ...

Super versions

$GL_{M|N}$, $OSP_{M|2N}$, periplectic, queer, ...



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Examples (that we will touch later)

Up to some slight change of setting we could also include:

Fusion categories or even finite additive Krull–Schmidt monoidal categories

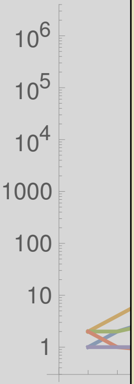
Proj$(G,\mathbb{K})$, **Inj**$(G,\mathbb{K})$, semisimpl. of quantum group reps, Soergel bimodules of finite type, ...

General additive Krull–Schmidt monoidal categories up to one condition (given later)

Rep(GL_n) and friends, quantum group reps, Soergel bimodules of affine type, ...

Most importantly, your favorite example might be included on this list

Let us not count!



im $V)^n$

lim $V)^n$ / n

lim $V)^n$ / n^2

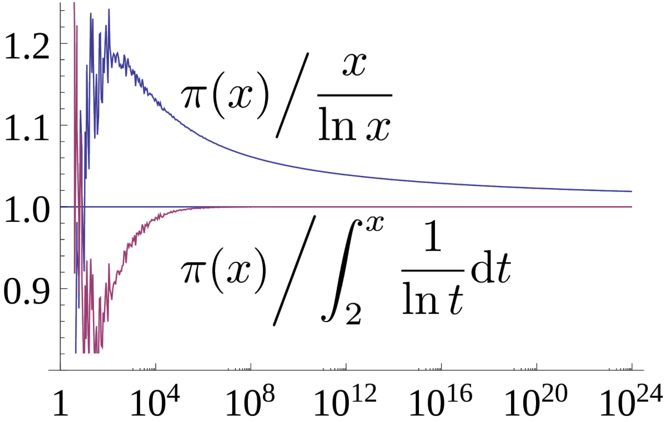
Let us pause for a second...the setting is way to general!

Decomposing $V^{\otimes n}$ for an arbitrary group is not happening

Better: Let us answer a different question!

- Γ = som
- $\mathbb{K}$ = any
- Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

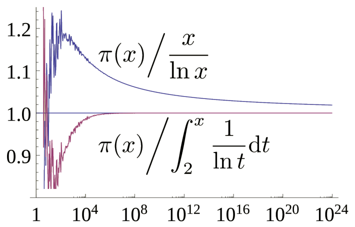
Let us not count!

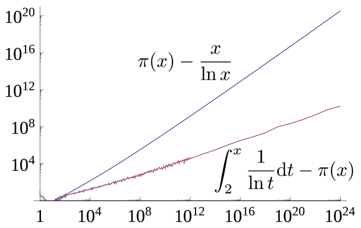


- Counting primes is difficult but...
- **Prime number theorem (many people ~1793)** #primes $= \pi(n) \sim n/\ln n$

Let us not count!

$\sim$ means asymptotically = ratios are good (not the absolute difference!)





So this is not doing the count!

- C
- **Prime number theorem (many people ~1793)** #primes = $\pi(n) \sim n/\ln n$

Let

Seriously, counting is difficult!

Legendre ∼1808:
(for $n/(\ln n - 1.08366)$)

Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

Acctually, #primes<1000 =1229...

Gauss, Legendre and company counted primes up to $n = 400000$ and more

That took years (your IPhone can do that in seconds...humans have advanced!)

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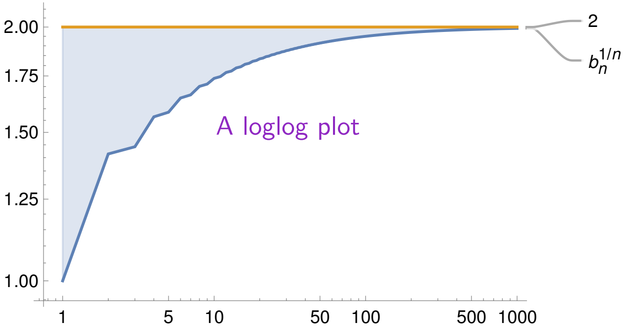
▶ Counting

▶ Prime n

"Discrete statements can often be solved approximately"

$(n) \sim n/\ln n$

Let us not count!

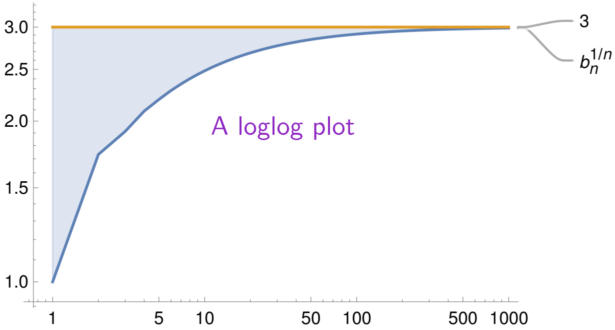


- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $2 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 1.99265$

Let us not count!



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- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathrm{Sym}\,\mathbb{C}^2$, then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $3 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 2.9875$

Let

Observation 1

Whatever is true for SL_2 over $\mathbb{C}$ is true in general, right?

So let us come back to the general setting:
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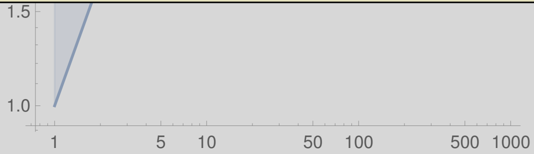

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Observation 2

$$b_n b_m \leq b_{n+m} \Rightarrow$$
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n}$$
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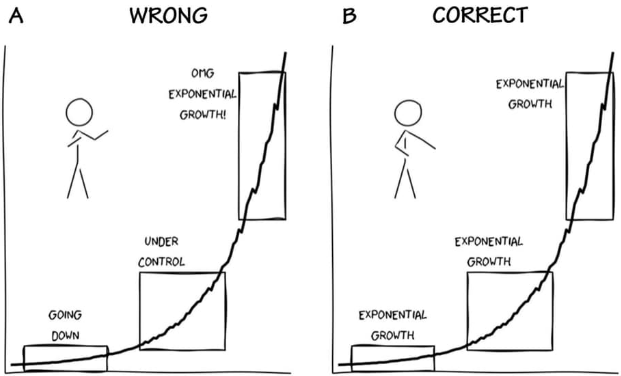
Observation 3

$1 \leq \beta \leq \dim_{\mathbb{K}} V$

$\beta = 1 \Leftrightarrow V^{\otimes n}$ for $n \gg 0$ is 'one block'

$\beta = \dim_{\mathbb{K}} V \Leftrightarrow$ summands of $V^{\otimes n}$ for $n \gg 0$ are 'essentially one-dimensional'

Let us not count!

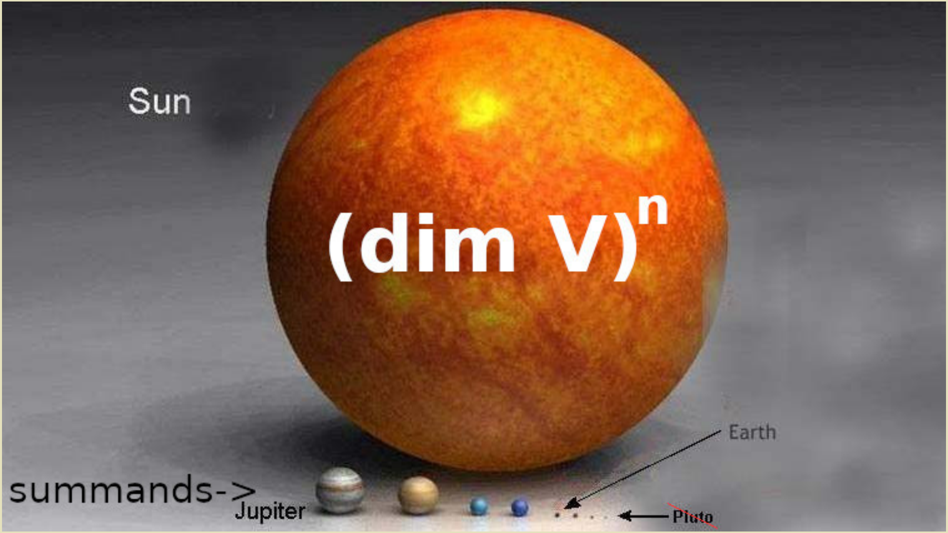


We have

$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

Exponential growth is scary

In other words, compared to the size of the exponential growth of $(\dim_{\mathbb{K}} V)^n$ all indecomposable summands are 'essentially one-dimensional'



Let us not count!

Tell you the honest asymptotics

Essentially what we have seen in

" $b_n \sim a_n \cdot (\dim_{\mathbb{K}} V)^n$ "

for a non-exponential a_n that is killed by $\sqrt[n]{\_}$

Tell you how to show this

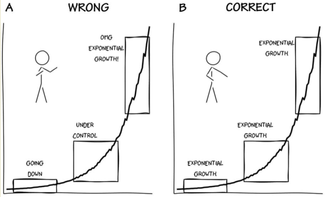
The argument is cute, I promise!

We have

$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

On the next slide there is a formula for

$$\underbrace{b_n}_{b(n)} \sim \underbrace{a_n \cdot (\dim_{\mathbb{K}} V)^n}_{a(n)} \text{''}$$



We will explore the formula by examples
so no need to memorize it

The take away messages are:

The formula is completely explicit

It only depends on eigenvalues and eigenvectors associated to a matrix

The assumptions on the next slide are not necessary
but make the formula look nicer

The finite case

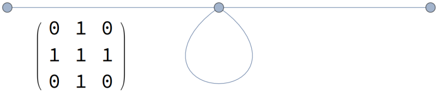
- Take a finite based $\mathbb{R}_{\geq 0}$-algebra R with basis $C = \{c_0, ..., c_{r-1}\}$
- Assume that R is the Grothendieck ring of our starting category
- For $a_i \in \mathbb{R}_{\geq 0}$, the action matrix M of $c = a_0 \cdot c_0 + ... + a_{r-1} \cdot c_{r-1} \in R$ is the matrix of left multiplication of c on C
- Assume that M has a leading eigenvalue λ of multiplicity one; all other eigenvalues of the same absolute value are $\exp(k2\pi i/h)\lambda$ for some h
- Denote the right and left eigenvectors of M for λ and $\exp(k2\pi i/h)\lambda$ by v_i and w_i, normalized such that $w_i^T v_i = 1$
- Let $v_i w_i^T[1]$ denote taking the sum of the first column of the matrix $v_i w_i^T$
- The formula $b(n) \sim a(n)$ we are looking for is

$$b(n) \sim \left(v_0 w_0^T[1] \cdot 1 + v_1 w_1^T[1] \cdot \zeta^n + v_2 w_2^T[1] \cdot (\zeta^2)^n + ... + v_{h-1} w_{h-1}^T[1] \cdot (\zeta^{h-1})^n\right) \cdot \lambda^n$$

- The convergence is geometric with ratio $|\lambda^{sec}/\lambda|$

The finite case

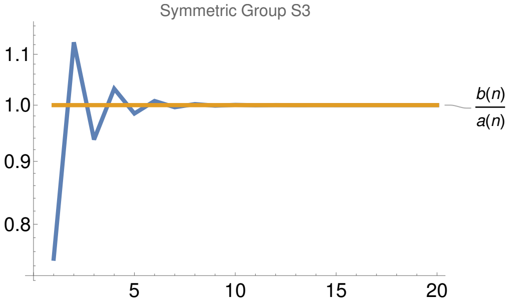
Symmetric group S_3, $\mathbb{K} = \mathbb{C}$, V=standard rep



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

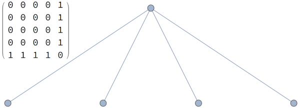
Example $\lambda = 2$, others$=0, -1$, $v = w = 1/\sqrt{6}(1,2,1)$, $vw^T = \begin{pmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{pmatrix}$ and

$$a(n) = \frac{2}{3} \cdot 2^n$$



The finite case

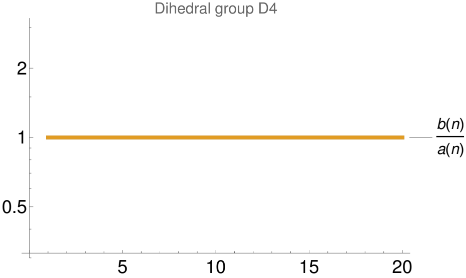
Dihedral group D_4 of order 8, $\mathbb{K} = \mathbb{C}$, V=defining rotation rep

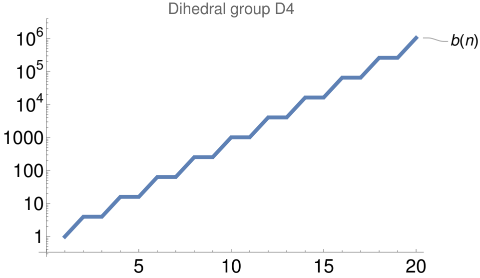


$$\begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 1&1&1&1&0 \end{pmatrix}$$

Example $\lambda = 2$, others$=-2, 0, 0, 0$, $v_\lambda = w_\lambda = 1/\sqrt{8}(1,1,1,1,2)$
$v_{-2} = w_{-2} = 1/\sqrt{8}(-1,-1,-1,-1,2)$ and

$$a(n) = \left(\tfrac{3}{4} + \tfrac{1}{4}(-1)^n\right) \cdot 2^n$$





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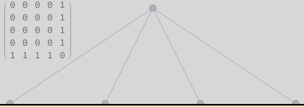
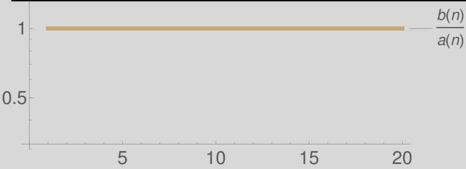


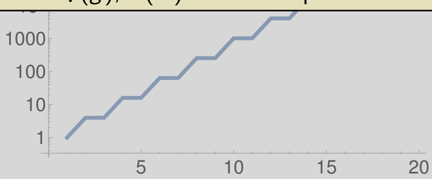
Example (general finite group, $\mathbb{K}=\mathbb{C}$, V=any faithful G-rep)

In this case we have a general formula:

$$a(n)=\left(\frac{1}{\#G}\sum_{g\in Z_V(G)}\Big(\sum_{L\in \mathrm{S}(G)}\omega_L(g)\dim_{\mathbb{C}}L\Big)\cdot\omega_V(g)^n\right)\cdot(\dim_{\mathbb{C}}V)^n$$

$Z_v(G)$=elements g acting by a scalar $w_V(g)$; $S(G)$=set of simples





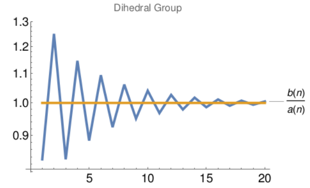
Example (continued)

Symmetric group S_m $a(n) = \left(\sum_{k=0}^{m/2} 1/((m-2k)!k!2^k)\right) \cdot \dim_{\mathbb{C}} V$

Dihedral group D_m of order $2m$

$$a(n) = \begin{cases} \frac{m+1}{2m} \cdot 2^n & \text{if } m \text{ is odd,} \\ \frac{m+2}{2m} \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is odd,} \\ \left(\frac{(m+2)}{2m} \cdot 1 + \frac{1}{m} \cdot (-1)^n\right) \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is even.} \end{cases}$$

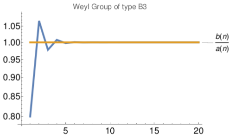
$m = 5:$



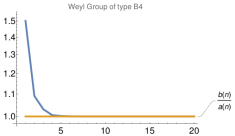
Complex reflection group $G(d,1,m)$

$\begin{cases} d = 1, \\ m = 3 \end{cases} : a(n) = \frac{2}{3} \cdot 3^n$, $\begin{cases} d = 2, \\ m = 3 \end{cases} : a(n) = \frac{5}{12} \cdot 3^n$, $\begin{cases} d = 2, \\ m = 4 \end{cases} : a(n) = \left(\frac{19}{96} \cdot 1 + \frac{1}{32} \cdot (-1)^n\right) \cdot 4^n$

$\begin{cases} d = 2, \\ m = 3 \end{cases} :$



, $\begin{cases} d = 2, \\ m = 4 \end{cases} :$



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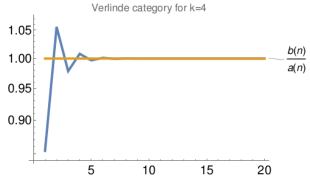
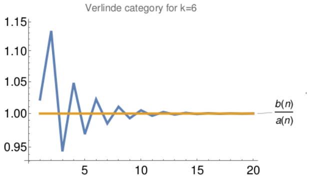
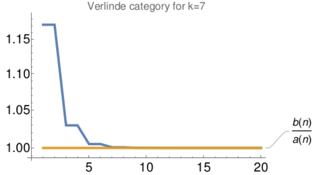
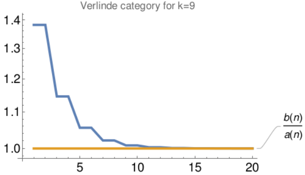
The finite case

$k=4:$ 

, $k=6:$ 

$k=7:$ 

, $k=9:$ 

Example For the SL_2 Verlinde category over $\mathbb{C}$ at level *k* and *V*=gen. object:

$$a(n)=\begin{cases}\frac{[1]_q+\dots+[k]_q}{[1]_q^2+\dots+[k]_q^2}\cdot\big(2\cos(\pi/(k+1))\big)^n & \text{if } k \text{ is even,}\\ \left(\frac{[1]_q+\dots+[k]_q}{[1]_q^2+\dots+[k]_q^2}\cdot 1+\frac{[1]_q-[2]_q+\dots-[k-1]_q+[k]_q}{[1]_q^2+\dots+[k]_q^2}\cdot(-1)^n\right)\cdot\big(2\cos(\pi/(k+1))\big)^n & \text{if } k \text{ is odd.}\end{cases}$$

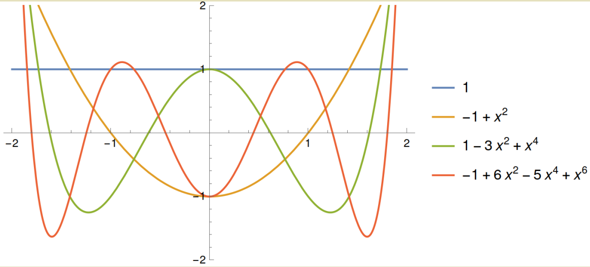
The finite ca

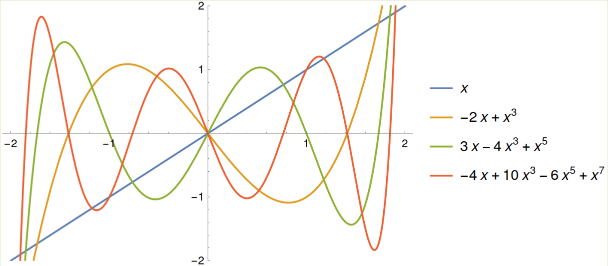
$k = 4$:

$k = 7$:

Example (continued)

The growth rate in this case is **not in $\mathbb{N}$**
but rather the leading root of the Chebyshev polynomial:





Example For … n. object:

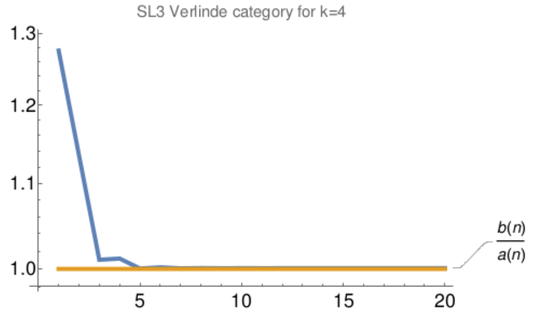
$a(n) = \begin{cases} \frac{[1]_q + \dots}{[1]_q^2 + \dots} & \text{if } k \text{ is even,} \\ \left(\frac{[1]_q + \dots}{[1]_q^2 + \dots}\right)^n & \text{if } k \text{ is odd.} \end{cases}$

Example (continued)

Here is the SL_3 Verlinde category over $\mathbb{C}$ at level $k=4$ and V=gen. object:

$$k=4\colon a(n)=\tfrac{1}{7}\Big(2+2\cos\big(\tfrac{3\pi}{7}\big)\Big)\cdot\Big(1+2\cos\big(\tfrac{2\pi}{7}\big)\Big)^n,$$

$k=4$:



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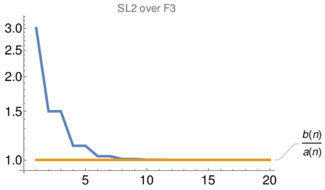
Koornwinder polynomials make their appearance

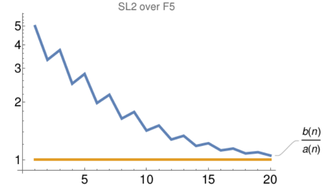
$\Big(\big(\frac{[1]_q+\dots+[k]_q}{[1]_q^2+\dots+[k]_q^2}\cdot 1+\frac{[1]_q-[2]_q+\dots-[k-1]_q+[k]_q}{[1]_q^2+\dots+[k]_q^2}\cdot(-1)^n\big)\cdot(2\cos(\pi/(k+1)))\Big)^n$ if k is odd.

The finite case

$p = 3$:



$p = 5$:

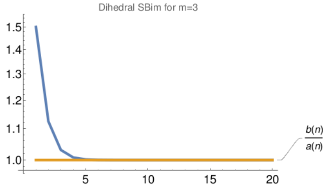


Example For $\mathrm{SL}_2(\mathbb{F}_p)$, $\mathbb{K} = \mathbb{F}_p$ and $V = \mathbb{F}_p^2$ we get:

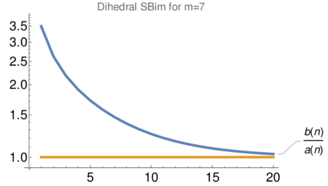
$$a(n) = \left(\frac{1}{2p-2} \cdot 1 + \frac{1}{2p^2 - 2p} \cdot (-1)^n \right) \cdot 2^n$$

The finite case

$m = 3$:



$m = 7$:

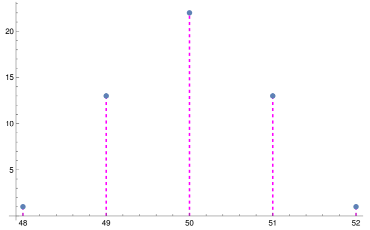


Example For dihedral Soergel bimodules of D_m, $\mathbb{K} = \mathbb{C}$ and $V = B_{st}$ we get:

$$a(n) = \frac{1}{2m} \cdot 4^n$$

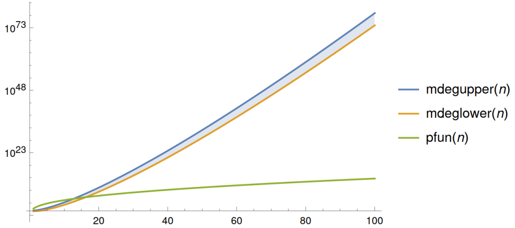
The finite case

The leading eigenvalue of 100 50-by-50 0-1-matrices :



- **Almost all** n-by-n 0-1-matrices have leading eigenvalue $\approx n/2$
- And indeed, for most categories the leading eigenvalue is large, e.g.

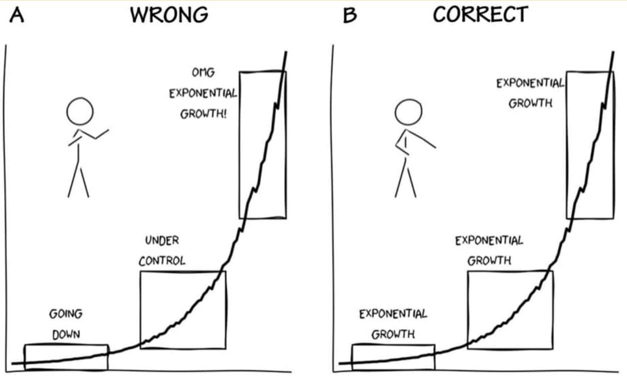
The max. leading eigenvalue for S_n : versus number of its simples



The finite case

So I somewhat just showed you the “wrong” examples

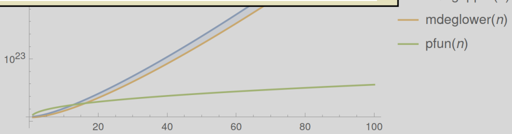
Anyway, observe that the growth of $b(n)$ is always exponential



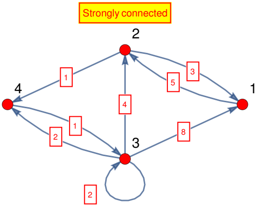
5

- Almos
- And in ...g.

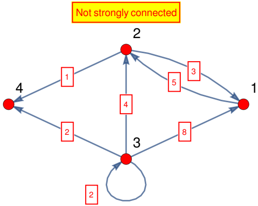
The
eigenvalue for S_n :
versus number of its simples



Eigenvalues and growth rates



$$\rightsquigarrow \begin{pmatrix} 0 & 3 & 8 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$



$$\rightsquigarrow \begin{pmatrix} 0 & 3 & 8 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

- One can and I will identify matrices and graphs
- Strongly connected = connected in the oriented sense

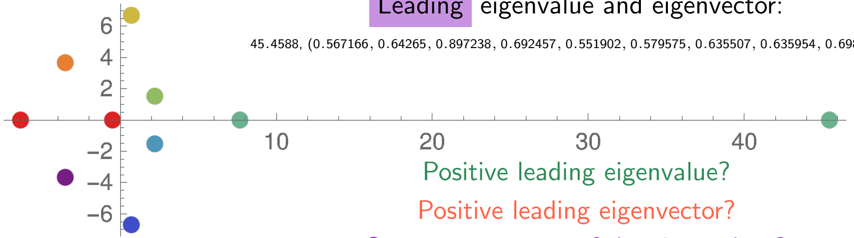
Eigenvalues and growth rates

$$
\begin{pmatrix}
3 & 0 & 5 & 1 & 8 & 7 & 0 & 1 & 4 & 7 \\
4 & 8 & 0 & 6 & 3 & 4 & 2 & 6 & 8 & 3 \\
8 & 6 & 6 & 7 & 6 & 0 & 9 & 4 & 8 & 5 \\
3 & 7 & 7 & 1 & 5 & 6 & 4 & 1 & 7 & 4 \\
4 & 0 & 3 & 4 & 4 & 8 & 8 & 1 & 4 & 2 \\
0 & 3 & 7 & 3 & 2 & 4 & 2 & 2 & 3 & 8 \\
6 & 3 & 6 & 1 & 5 & 6 & 1 & 6 & 4 & 4 \\
2 & 4 & 0 & 2 & 8 & 8 & 1 & 4 & 8 & 6 \\
6 & 7 & 6 & 3 & 4 & 2 & 9 & 6 & 5 & 0 \\
0 & 6 & 9 & 9 & 8 & 3 & 9 & 9 & 1 & 9
\end{pmatrix}
$$

What on earth is going on? Strange patterns with the eigenvalues and vectors:

Leading eigenvalue and eigenvector:

45.4588, (0.567166, 0.64265, 0.897238, 0.692457, 0.551902, 0.579575, 0.635507, 0.635954, 0.698596, 1)



Positive leading eigenvalue?

Positive leading eigenvector?

Strange symmetry of the eigenvalues?

Eig

Non-negativity is key!

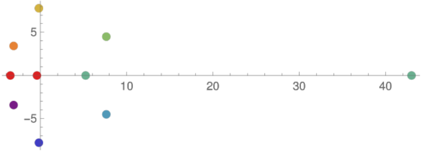
Non-negative. The pattern persist:

$$\begin{pmatrix} 1 & 3 & 8 & 6 & 6 & 2 & 3 & 6 & 8 & 7 \\ 5 & 5 & 1 & 8 & 3 & 0 & 3 & 3 & 7 & 6 \\ 0 & 1 & 6 & 3 & 6 & 7 & 5 & 3 & 9 & 0 \\ 1 & 0 & 2 & 8 & 7 & 2 & 8 & 8 & 3 & 9 \\ 9 & 2 & 6 & 1 & 9 & 6 & 3 & 2 & 6 & 5 \\ 6 & 7 & 0 & 1 & 4 & 5 & 9 & 0 & 4 & 5 \\ 0 & 4 & 8 & 4 & 1 & 4 & 0 & 2 & 5 & 6 \\ 6 & 6 & 1 & 5 & 7 & 4 & 2 & 7 & 3 & 0 \\ 7 & 7 & 3 & 0 & 2 & 0 & 6 & 4 & 8 & 2 \\ 6 & 7 & 9 & 3 & 1 & 2 & 1 & 8 & 7 & 4 \end{pmatrix} \rightsquigarrow$$

Leading eigenvalue and eigenvector:

42.9948

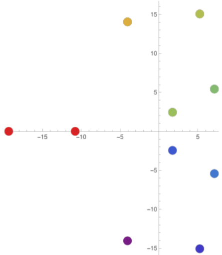
(1.05024, 0.889464, 0.800641, 1.02602, 1.05438, 0.850121, 0.704574, 0.893187, 0.796941, 1)



Wha

Negative. The pattern breaks:

$$\begin{pmatrix} -4 & 0 & 1 & -2 & 0 & -5 & 8 & 6 & 8 & 3 \\ -9 & -9 & 7 & 5 & 6 & 8 & -6 & 5 & 1 & 1 \\ 8 & 3 & -4 & -3 & -9 & 4 & -8 & -8 & -6 & 7 \\ 0 & -4 & -4 & -4 & -4 & 5 & 3 & -4 & 5 & -7 \\ 0 & 3 & -2 & 2 & 5 & 1 & -2 & 0 & 9 & 8 \\ 6 & 8 & 0 & -6 & -7 & 3 & -7 & -9 & -4 & -4 \\ -8 & 8 & 5 & 6 & -1 & 3 & 0 & -3 & -3 & 0 \\ 4 & 3 & -1 & -9 & 6 & -4 & 2 & -3 & -1 & 7 \\ -2 & 6 & 2 & -6 & -8 & -4 & -5 & 0 & 2 & -1 \\ -6 & -1 & -1 & 5 & -7 & 7 & 4 & 4 & 9 & 4 \end{pmatrix} \rightsquigarrow$$



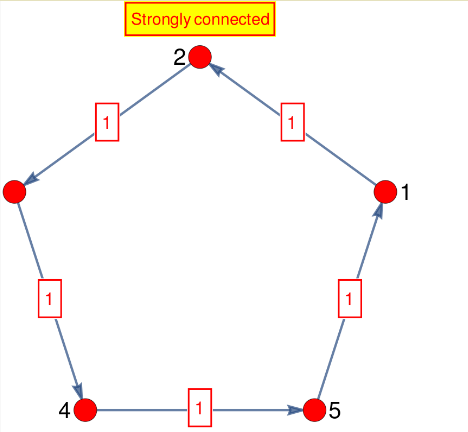
96, 1)

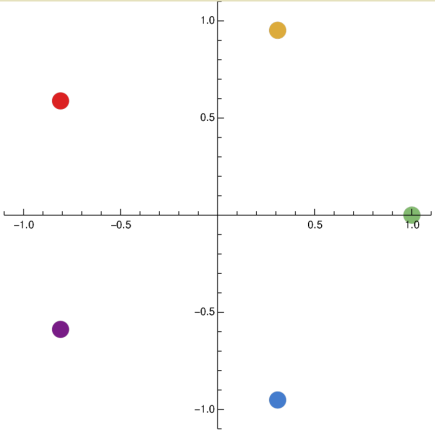
Strange symmetry of the eigenvalues?

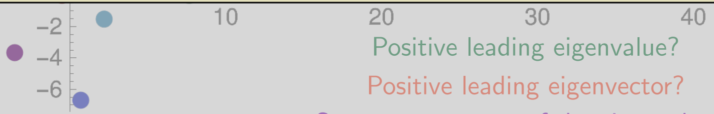
Eigenvalues and growth rates

$\begin{pmatrix} 3 & 0 & 5 & 1 & 8 & 7 & 0 & 1 & 4 & 7 \end{pmatrix}$

One more example we need to know:





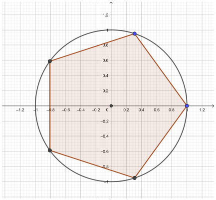


Positive leading eigenvalue?

Positive leading eigenvector?

Strange symmetry of the eigenvalues?

Eigenvalues and growth rates



Theorem (Perron–Frobenius ∼1907, Rothblum ∼1981) for $M \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$

- M has a leading eigenvalue λ; all other eigenvalues with $|\mu| = \lambda$ are precisely the vertices of a h_i-regular polygon of radius λ
- There is one such h_i-polygon for i from one to the multiplicity of λ
- Take $h = lcd(h_i)$. Then there exist (explicit) polynomials $S^i(n)$ such that

$$\lim_{n\to\infty} |\,(M/\lambda)^{hn+i} - S^i(n)| \to 0 \quad \forall i \in \{0, ..., h-1\},$$

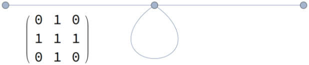
and the convergence is geometric with ratio $|\lambda^{sec}/\lambda|^h$

Eigenvalue

This shows all the stuff from the previous section!

The finite case

Symmetric group S_3, $\mathbb{K}=\mathbb{C}$, V=standard rep

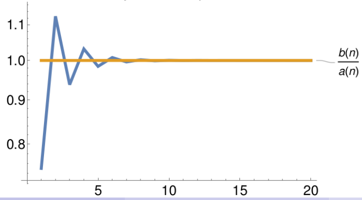


$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Example $\lambda = 2$, others=0, -1, $v = w = 1/\sqrt{6}(1,2,1)$, $vw^T = \begin{pmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{pmatrix}$ and

$$a(n) = \tfrac{2}{3}\cdot 2^n$$

Symmetric Group S3



We also get

$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

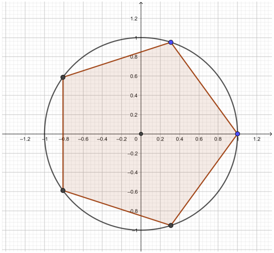
whenever the situation is finite and dim=PFdim

Theorem (... $\mathrm{Mat}_m(\mathbb{R}_{\geq 0})$

- M has ... re precisely the ve
- There ... λ
- Take ... ch that

and the convergence is geometric with ratio $|\lambda^{sec}/\lambda|^h$

Eigenvalues and growth rates



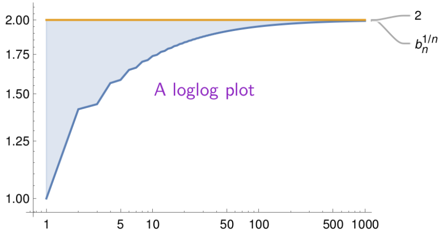
Theorem (Vere-Jones+others ∼1967) for $M \in \mathrm{Mat}_{\mathbb{N}}(\mathbb{R}_{\geq 0})$

- M has a leading eigenvalue $\lambda \in \mathbb{R}_{\geq 0} \cup \{\infty\}$
- If $\lambda < \infty$, then the polygon part is the same as before
- $(M^k)_{ij}$ growth $\leq$ exponentially $\Leftrightarrow \lambda < \infty$
- If $\lambda < \infty$ then $(M^k)_{ij} \cong a_n \lambda^n$ with non-exponential a_n
- If M is positively recurrent, then the approximation formula is as before
- The eigenvectors and eigenvalues can be approximated using cut-offs of M

Eigenvalues and growth rates

This gives approximation formulas in general!

Let us not count!



- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $2 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 1.99265$

We get

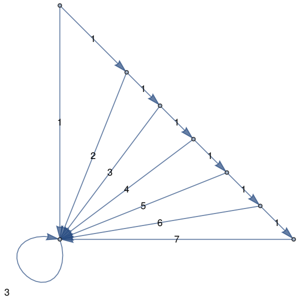
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

whenever dim=PFdim

Theorem (

- M has
- If $\lambda <$
- $(M^k)_{ij}$
- If $\lambda <$
- If M is ... efore
- The ei... ffs of M

The infinite case

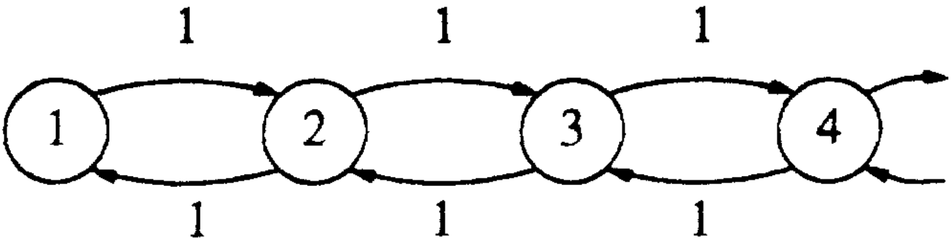


Example $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{K} = \overline{\mathbb{F}_2}$ and V=3 dim. indecomposable we get:

- Everything works: i.e. we have a finite $\lambda = 3$ and eigenvectors
- The growth rate is

$$a(n) = 3^n \Rightarrow \beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

The infinite case

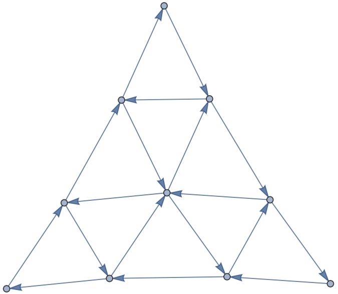


Example $\Gamma = \mathrm{SL}_2(\mathbb{C})$, $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^2$:

- We have $\lambda = 2$ but the eigenvectors are messed-up
- The growth rate is

$$a(n) = \underbrace{a_n}_{\text{sub. exp.}} \cdot 2^n \Rightarrow \beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

The infinite case



Example $\Gamma = \mathrm{SL}_3(\mathbb{C})$, $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^3$:

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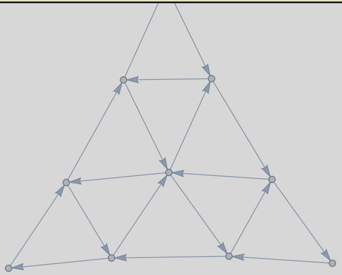
$$a(n) = \underbrace{a_n}_{\text{sub. exp.}} \cdot 3^n \Rightarrow \beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

The infinite c

Example The $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_3(\mathbb{C})$ examples generalize...

...to include arbitrary (faithful) fdim reps

...to other connected reductive algebraic groups



Example $\Gamma = \mathrm{SL}_3(\mathbb{C})$, $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^3$:

- We have $\lambda = 3$ but the eigenvectors are messed-up
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$$a(n) = \underbrace{a_n}_{\text{sub. exp.}} \cdot 3^n \Rightarrow \beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

The infinite c

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Example The $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_3(\mathbb{C})$ examples generalize...

...to include arbitrary (faithful) fdim reps

...to other connected reductive algebraic groups

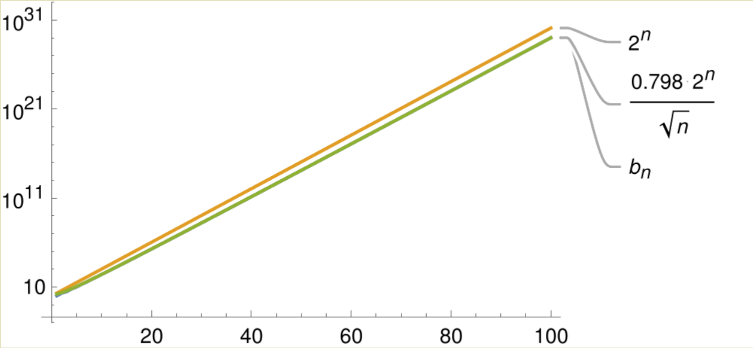
Example A bit more work recovers the **Coulembier–Etingof–Ostrik formula ∼2023**:

$$a(n) = s_V(n) n^{-\#\text{pos. roots}/2} \cdot (\dim_{\mathbb{C}} V)^n$$

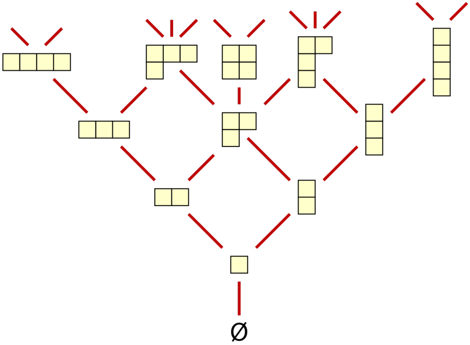
for an explicit $s_V(n)$

Example ($\mathrm{SL}_2(\mathbb{C})$, $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^2$)

$$a(n) = \sqrt{2/\pi}\, n^{-1/2} \cdot 2^n \approx 0.798 n^{-1/2} \cdot 2^n$$



The infinite case

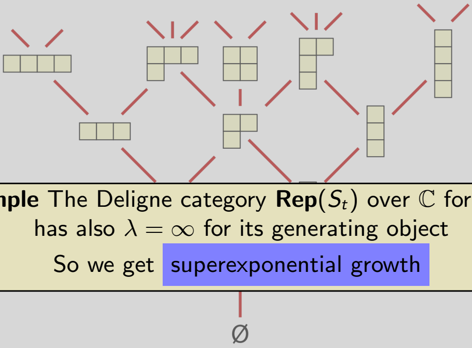


Example $\Gamma = \mathrm{GL}_{\mathbb{N}}(\mathbb{C})$, $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^{\mathbb{N}}$:

- We have $\lambda = \infty$ and the eigenvectors are messed-up
- The growth rate is thus

superexponential

The infinite case



Example The Deligne category $\mathbf{Rep}(S_t)$ over $\mathbb{C}$ for $t \notin \mathbb{Z}$ has also $\lambda = \infty$ for its generating object

So we get superexponential growth

Example $\Gamma = \mathrm{GL}_{\mathbb{N}}(\mathbb{C})$, $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^{\mathbb{N}}$:

- We have $\lambda = \infty$ and the eigenvectors are messed-up
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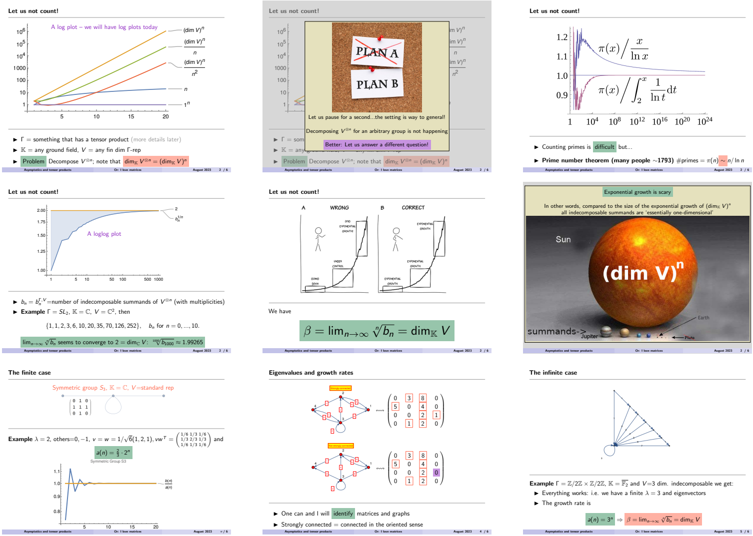
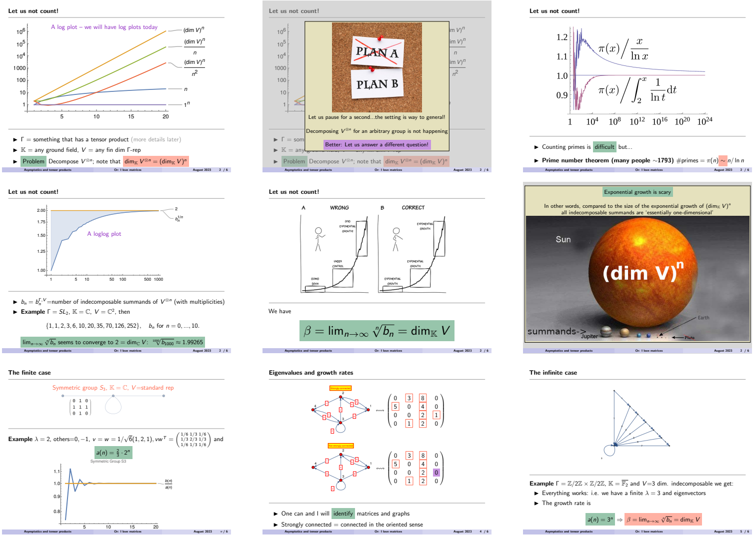


There is still much to do...



Thanks for your attention!