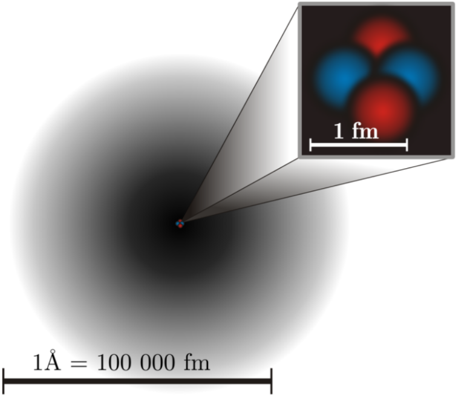
Matrices and moduli

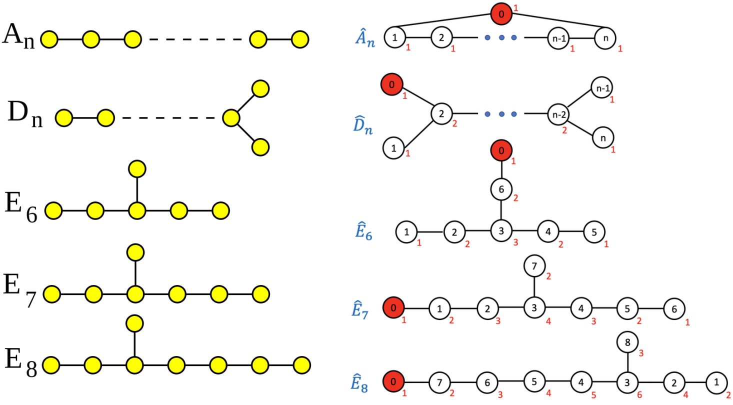
Or: Almost all = boring?

~~Accept~~Change what you cannot ~~change~~accept



August 2023

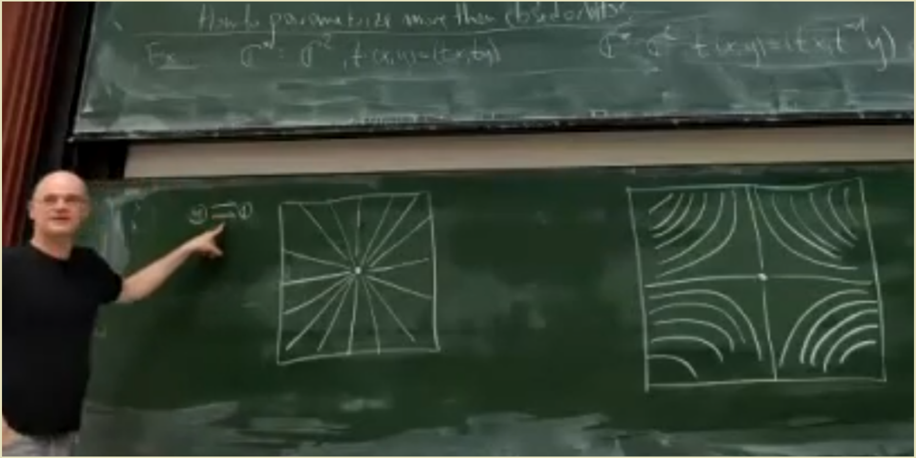
Matrix problems – the algebraic approach



- Recall Some matrix problems can be associated with quivers
- Recall Matrix problems are doable only in the finite and affine ADE types
- Otherwise, the algebraic approach is doomed to fail and classifications get wild

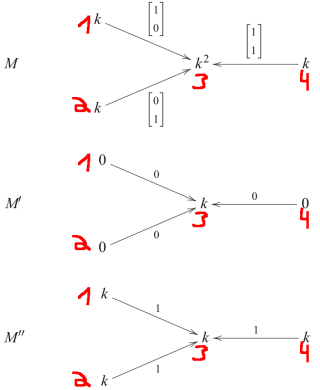
Today

A geometric approach to matrix problems
following Reinecke's Felix Klein lecture 2020 (ask Dr. google for 5 brilliant video lectures)



But first let me wrap-up the algebraic approach

Matrix problems – the algebraic approach



$\boldsymbol{d}(M) = (1, 1, 2, 1)$

, $\boldsymbol{d}(M') = (0, 0, 1, 0)$

$\boldsymbol{d}(M'') = (1, 1, 1, 1)$

- The classification of inde. is hopeless in general
- But for almost all inde. the classification is actually pretty easy
- We will see this momentarily dimension vector $\boldsymbol{d}$ wise

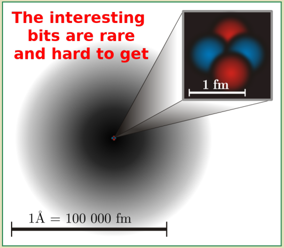
Matrix proble

General phenomena

“Really difficult”

often means

“easy almost all of the time, but hard for some cases”



I will show you now a fun example of this phenomena!

The example is not related to quivers
but this is how I learned this stuff ;-)
and we go back to quivers afterwards

- The classi
- But for al y
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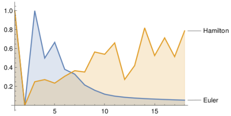
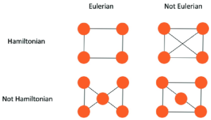
Matrix problems – the algebraic approach

Almost all (random) graphs are Hamiltonian; almost no (random) graph is Eulerian



- Hamiltonian = has a cycles that visits all vertices; Eulerian = has a cycles that visits all edges; looks similar, but is different:

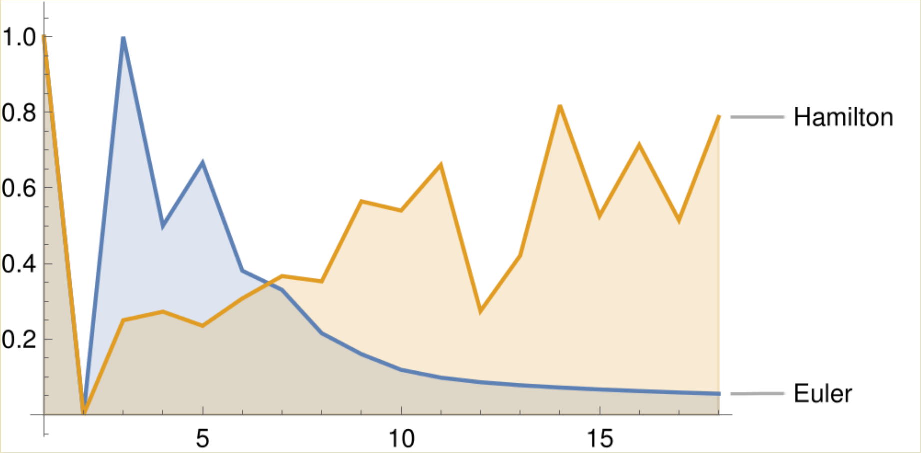


- Crucial (Almost all $\neq$ all) and (almost no $\neq$ no)!

- Checking whether a graph is Hamiltonian is NP complete = difficult as hell
- But for almost all graphs there are efficient algorithm to check this
- So the difficulty is very concentrated

Matrix problems – the algebraic approach

Almost all graphs are Hamiltonian:



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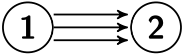
Matrix problems – the algebraic approach

Triple Kronecker: $K^3 =$



Kronecker's normal form for $(A, B) \approx (A', B')$:

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}, \quad id_n = \begin{pmatrix} 1 & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$L_n = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{pmatrix}, \quad L_n^T = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{pmatrix}^T$$

- Take $\boldsymbol{d} = (n, n)$ for K^3
- Assume that A is invertible, B is diagonalizable with pairwise different eigenvalues
- Using Kronecker's normal form we can assume that $A = id_n$ and $B = diag(\lambda_1, ..., \lambda_n)$

Matrix problems – the algebraic approach

Kronec

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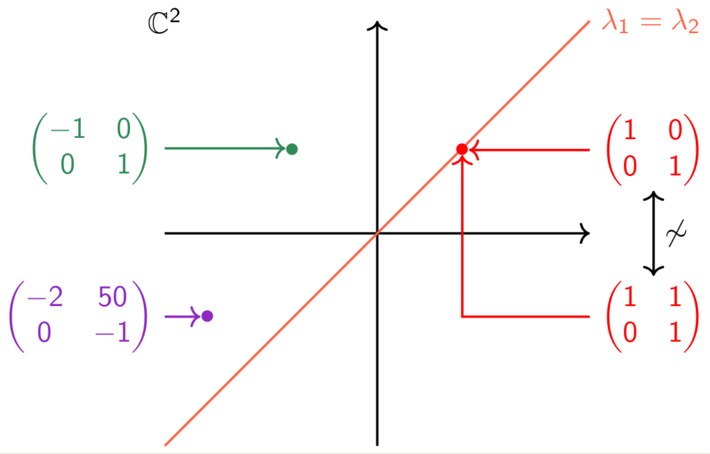
Almost all matrices have n different eigenvalues:



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- Take
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Matrix problems – the algebraic approach

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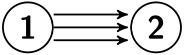

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- The subgroup $H \subset GL_n \times GL_n$ fixing (A, B) consists of diagonal matrices $diag(h_1, ..., h_n)$ acting on C by conjugation: $c_{i,j} \mapsto h_i/h_j \cdot c_{i,j}$
- Thus, we can assume that

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n-1} & c_{1,n} \\ 1 & c_{2,2} & \dots & c_{2,n-1} & c_{2,n} \\ c_{3,1} & 1 & \dots & c_{3,n-1} & c_{3,n} \\ & & \dots & & \\ c_{n,1} & c_{n,2} & \dots & 1 & c_{1,n} \end{pmatrix}, \quad c_{i,j} \neq 0$$

Theorem (folklore ∼1970s)

Almost all inde. K^3-reps with dimension (vector) (n,n)
are of the form $\big(id_n, diag(\lambda_1..., \lambda_n), C\big)$ as in the background

A bit more effort shows something similar for other dimensions vectors and quivers

The bait It is often very easy to classify almost all indecomposables

M

$\lambda)$ $1)$

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Kronecker's norm

$(A, B) \approx (A$

1

$0 \quad 1$

$\ddots \quad \ddots$

$0 \quad 1$

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T

This is the ignore the black sheep strategy

are very successful in all of math and even life

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This is the ignore the black sheep strategy

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- The subgroup … agonal matrices $diag(h_1, ..., h_n)$ … $c_{i,j}$
- Thus

The catch This "generic" classification kills a lot

Example

For the Jordan quiver $\textcircled{1}\circlearrowleft$ a generic classification reduces to diagonalizable matrices completely missing the Jordan normal form

Matrix problems – the algebraic approach

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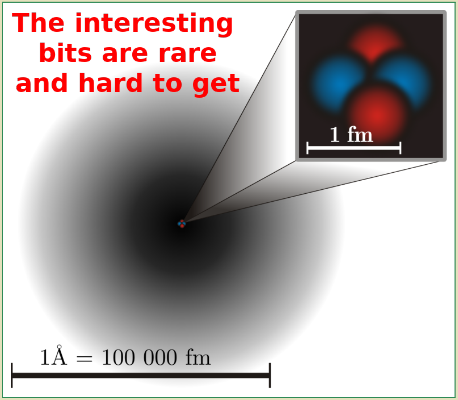

The algebraic approach get us to the empty space of the atom

The geometric approach should get us a bit closer to the interesting bits

- T … es
  d
- T

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n-1} & c_{1,n} \\ 1 & c_{2,2} & \dots & c_{2,n-1} & c_{2,n} \\ c_{3,1} & 1 & \dots & c_{3,n-1} & c_{3,n} \\ & & \dots & & \\ c_{n,1} & c_{n,2} & \dots & 1 & c_{1,n} \end{pmatrix}, \quad c_{i,j} \neq 0$$

Matrix

For completeness (I will not recall what root systems are)

Theorem (Kac∼1980)
For an arbitrary quiver we only have two cases:

(a) If $\boldsymbol{d}$ is a positive real root, then $!\exists$ inde. rep. with dimension $\boldsymbol{d}$

(b) If $\boldsymbol{d}$ is a positive imaginary root, then $\exists$ inde. rep. with dimension $\boldsymbol{d}$ parametrized by $1-1/2(\boldsymbol{d},\boldsymbol{d})$ parameters

Kron

$(A,B)\approx(A',B')$

Example (type D_4)
SageMath with Phi = RootSystem(['D',4]).root_poset(); produces:

The 3-subspace problem is of finite representation type (D_4); the indecomposables are (up to "permutation of legs"):

$$\begin{tikzcd}
0 \arrow[r, "0"] & \mathbf{k} & 0 \arrow[l, "0"'] \\
& 0 \arrow[u, "0"] &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{k} \arrow[r, "0"] & 0 & 0 \arrow[l, "0"'] \\
& 0 \arrow[u, "0"] &
\end{tikzcd}
\quad
\begin{tikzcd}
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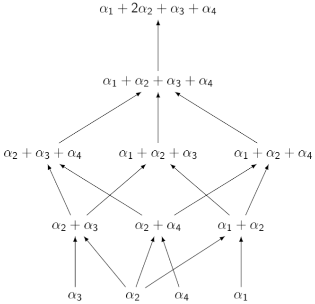
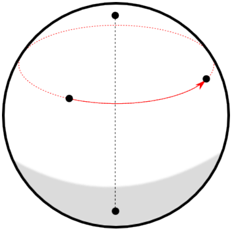
& \mathbf{k} \arrow[u, "1"] &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{k} \arrow[r, "\binom{1}{0}"] & \mathbf{k}\oplus\mathbf{k} & \mathbf{k} \arrow[l, "\binom{0}{1}"'] \\
& \mathbf{k} \arrow[u, "\binom{1}{1}"] &
\end{tikzcd}$$



Moduli spaces – semisimple case



- Basic idea Fix $\boldsymbol{d}$, and M of dimension $\boldsymbol{d}$, and consider the affine $\mathbb{C}$-space

$$R_{\boldsymbol{d}} = R_{\boldsymbol{d}}(Q) = \bigoplus_{i \to j} \mathrm{hom}_{\mathbb{C}}(M_i, M_j)$$

  $G_{\boldsymbol{d}} = \prod_i GL(M_i)$ acts on $R_{\boldsymbol{d}}$ via base change, and $G_{\boldsymbol{d}}$-orbits correspond bijectively to the iso. classes of Q-reps of dimension $\boldsymbol{d}$

- Task Find a subset $U \subset R_{\boldsymbol{d}}$, an algebraic variety X and a morphism $\pi\colon U \to X$ whose fibers are precisely the $G_{\boldsymbol{d}}$-orbits in U

Moduli spaces – semisimple case

Problem

Take the 5-Kronecker quiver  and $M(\lambda,\mu)$ for $(\lambda,\mu)\neq(0,0)$ and $\boldsymbol{d}=(2,3)$:

$$\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\\lambda&0\\0&0\end{bmatrix},\begin{bmatrix}0&\mu\\0&0\\0&0\end{bmatrix}$$

Lemma (easy) $M(\lambda,\mu)\cong M(\alpha,\beta)$ if and only if $\exists t\in\mathbb{C}^*$ such that $\lambda=t\alpha$ and $\mu=t^{-1}\beta$

Let U be the set of all $M(\lambda,\mu)$

Then $\lim_{\lambda\to 0}M(\lambda,1)=M(0,1)$ and $\lim_{\mu\to 0}M(1,\mu)=M(1,0)$ in U

Hence, there can not be a continuous map $\pi\colon U\to X$ since

$M(\lambda,1)\cong M(1,\mu)$ but $M(0,1)\not\cong M(1,0)$

bijectively to the iso. classes of Q-reps of dimension $\boldsymbol{d}$

- Task Find a subset $U\subset R_{\boldsymbol{d}}$, an algebraic variety X and a morphism $\pi\colon U\to X$ whose fibers are precisely the $G_{\boldsymbol{d}}$-orbits in U

Moduli spaces – semisimple case

Problem

Take the 5-Kronecker quiver  and $M(\lambda, \mu)$ for $(\lambda, \mu) \neq (0,0)$ and $\boldsymbol{d} = (2,3)$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \lambda & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mu \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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bijectively to the iso. classes of Q-reps of dimension $\boldsymbol{d}$

- The above example is just one of the typical problems in defining quotients: it shows that the potential "orbit space $U/G_{\boldsymbol{d}}$" would be non-separated Usually set-theoretical quotients have a bad topology – need something better!

$\pi \colon U \to X$ whose fibers are precisely the $G_{\boldsymbol{d}}$ orbits in U

Moduli spaces – semisimple case

$$\begin{tikzcd} R_{\boldsymbol{d}} \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X \arrow[ur, dashed, "\exists ! \overline{f}"'] & \end{tikzcd}$$

- $G_{\boldsymbol{d}}$ acts on $R_{\boldsymbol{d}}$ as before, (X, π) should be universal
- In the cat. of sets with a $G_{\boldsymbol{d}}$-action we get the “bad” quotient $X = R_{\boldsymbol{d}}/G_{\boldsymbol{d}}$,

  in the cat. of alg. varieties with a $G_{\boldsymbol{d}}$-action we get the “better” quotient $X = R_{\boldsymbol{d}}//G_{\boldsymbol{d}}$
- **Theorem ((Hilbert–)Mumford ~(1893,)1965)** $R_{\boldsymbol{d}}//G = Spec(\mathbb{C}[R_{\boldsymbol{d}}]^{G_{\boldsymbol{d}}})$ and parametrizes the closed orbits

Moduli spaces – semisimple case

$$\begin{tikzcd}
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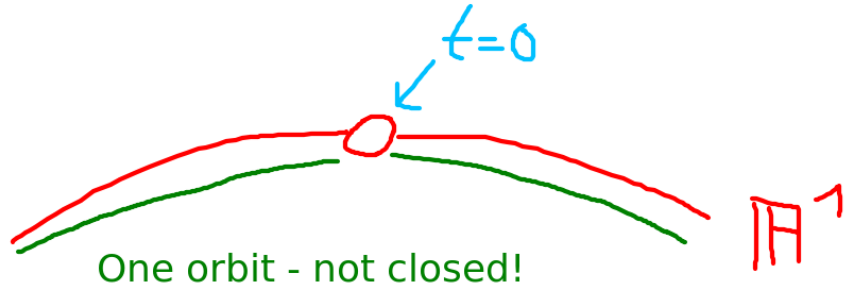
\end{tikzcd}$$

- **Theorem (Le Bruyn–Procesi ∼1990)** $R_{\boldsymbol{d}}//G = Spec(\mathbb{C}[R_{\boldsymbol{d}}]^{G_{\boldsymbol{d}}})$ and parametrizes iso. classes of semisimple Q-reps of dimension $\boldsymbol{d}$
- Closed orbit ⇔ semisimple
- Call $M^{ss}_{\boldsymbol{d}} = R_{\boldsymbol{d}}//G$ the moduli space of semisimple Q-reps of dimension $\boldsymbol{d}$

Moduli spaces – semisimple case

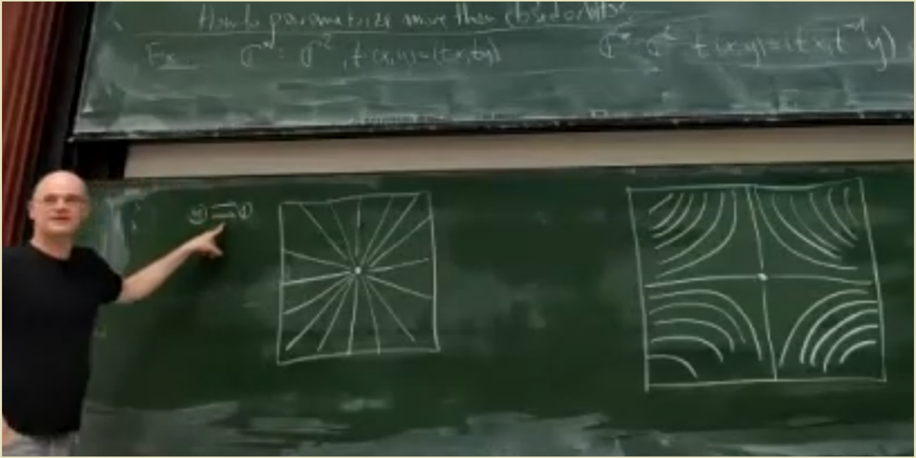
"Proof" of **Closed orbit ⇔ semisimple**

If a Q-rep of dimension 2 of ①⟲ is not semisimple
then we can assume that we have the matrix $\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}$
For $t \neq 0$ this is a nontrivial Jordan block up to base change
For $t = 0$ this is a direct sum of two $1d$ simples
Thus, the orbit of the nontrivial Jordan block is not closed and looks like



- Call $M_{\boldsymbol{d}}^{ss} = R_{\boldsymbol{d}}//G$ the moduli space of semisimple Q-reps of dimension $\boldsymbol{d}$

Example (left: ① $\rightrightarrows$ ②, right: ① $\rightleftarrows$ ②)



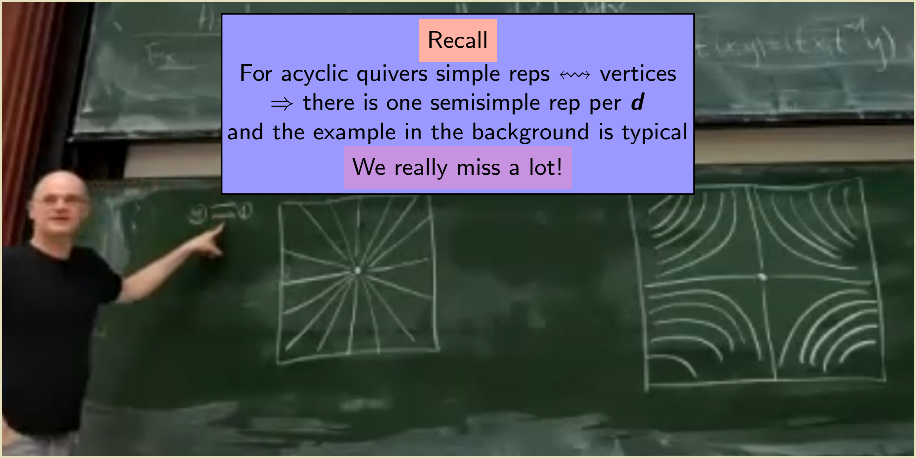
For $\boldsymbol{d} = (1,1)$, the action of $G_{\boldsymbol{d}} \cong \mathbb{C}^*$ is $t(x,y) = (tx,ty)$ and $t(x,y) = (tx,t^{-1}y)$

The orbit spaces are as above

The closed orbits are $(0,0)$; plus hyperbolas on the right

We miss a lot!

Example (left: ①$\rightrightarrows$②, right: ①$\rightleftarrows$②)



Recall

For acyclic quivers simple reps $\leftrightsquigarrow$ vertices

$\Rightarrow$ there is one semisimple rep per $\boldsymbol{d}$

and the example in the background is typical

We really miss a lot!

For $\boldsymbol{d} = (1,1)$, the action of $G_{\boldsymbol{d}} \cong \mathbb{C}^*$ is $t(x,y) = (tx, ty)$ and $t(x,y) = (tx, t^{-1}y)$

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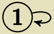
We miss a lot!

Moduli spaces – semisimple case

Example (Jordan quiver )

$\boldsymbol{d} = (2)$, $G_{\boldsymbol{d}} = GL_2(\mathbb{C})$ acting on $R_{\boldsymbol{d}} = Mat_2(\mathbb{C})$ by conjugation
There are "obvious" $GL_2(\mathbb{C})$-invariant functions:
the trace $tr(\_)$ and the determinant $det(\_)$

Lemma $\mathbb{C}[Mat_2(\mathbb{C})]^{GL_2(\mathbb{C})}$ is generated by $tr(\_)$ and $det(\_)$

Lemma $tr(\_)$ and $det(\_)$ are algebraically independent

Hence, $\mathbb{C}[Mat_2(\mathbb{C})]^{GL_2(\mathbb{C})} \cong \mathbb{C}[X, Y]$ and $Mat_2(\mathbb{C})//GL_2(\mathbb{C})$ is affine 2-space

- **Theorem (Le Bruyn–Procesi ~1990)** $R_{\boldsymbol{d}}//G = Spec(\mathbb{C}[R_{\boldsymbol{d}}]^{G_{\boldsymbol{d}}})$ and parametrizes iso. classes of semisimple Q-reps of dimension $\boldsymbol{d}$
- Closed orbit ⇔ semisimple
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Moduli spaces – semisimple case

Example (Jordan quiver ①↺)

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Hence, $\mathbb{C}[Mat_2(\mathbb{C})]^{GL_2(\mathbb{C})} \cong \mathbb{C}[X, Y]$ and $Mat_2(\mathbb{C})//GL_2(\mathbb{C})$ is affine 2-space

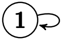
Example (Jordan quiver ①↺ - second)

For $\boldsymbol{d} = (n)$ we have that $\mathbb{C}[Mat_n(\mathbb{C})]^{GL_n(\mathbb{C})} = \mathbb{C}[e_i(\_)|i = 1, ..., n]$
The $e_i(\_)$ are the coefficients of the characteristic polynomial

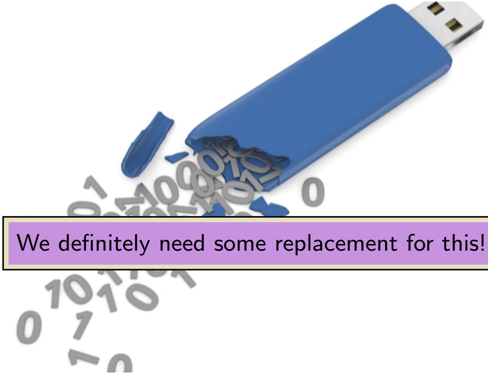
Alternatively, a diagonalizable matrix mod base change is determined by its eigenvalues!

- Call $M^{ss}_{\boldsymbol{d}} = R_{\boldsymbol{d}}//G$ the moduli space of semisimple Q-reps of dimension $\boldsymbol{d}$

Moduli spaces – semisimple case

- **Theorem (Le Bruyn–Procesi ∼1990)** $\mathbb{C}[R_{\boldsymbol{d}}]^{G_{\boldsymbol{d}}}$ is generated by “traces along oriented cycles”
- Problem 1 The theory is trivial for quivers without oriented cycles
- Problem 2 In general, we loose a lot, e.g. the Jordan normal form for 

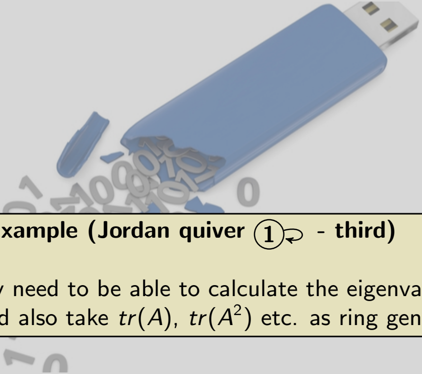
Moduli spaces – semisimple case



We definitely need some replacement for this!

- **Theorem (Le Bruyn–Procesi ∼1990)** $\mathbb{C}[R_{\boldsymbol{d}}]^{G_{\boldsymbol{d}}}$ is generated by "traces along oriented cycles"
- Problem 1 The theory is trivial for quivers without oriented cycles
- Problem 2 In general, we loose a lot, e.g. the Jordan normal form for ①⟲

Moduli spaces – semisimple case



Example (Jordan quiver ①⟲ - third)

We only need to be able to calculate the eigenvalues so we could also take $tr(A)$, $tr(A^2)$ etc. as ring generators

- ▶ **Theorem (Le Bruyn–Procesi ∼1990)** $\mathbb{C}[R_{\boldsymbol{d}}]^{G_{\boldsymbol{d}}}$ is generated by “traces along oriented cycles”
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Moduli sp

Example ()

For $\boldsymbol{d} = (2)$ one can show that $\mathbb{C}[Mat_n(\mathbb{C}) \times Mat_n(\mathbb{C})]^{GL_n(\mathbb{C})}$ is generated by $tr(A)$ $tr(A^2)$, $tr(AB) = tr(BA)$, $tr(B)$ and $tr(B^2)$

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Example ($\circlearrowleft ① \circlearrowright$)

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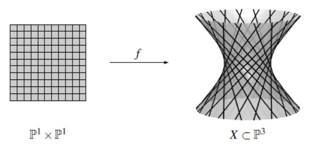
Example ($\begin{tikzcd} ① \arrow[r, shift left, "s"] & ② \arrow[l, shift left, "t"] \end{tikzcd}$ with s and t arrows)

For $\boldsymbol{d} = (1,1)$ one can show that the invariant ring is generated by $tr(s_i t_j) = s_i t_j$

Moreover, $R_{\boldsymbol{d}}//G_{\boldsymbol{d}} \cong Cone(\mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \hookrightarrow \mathbb{P}^{s+t-1})$ (via Segre embedding)

Example (Segre embedding for $s = t = 2$)

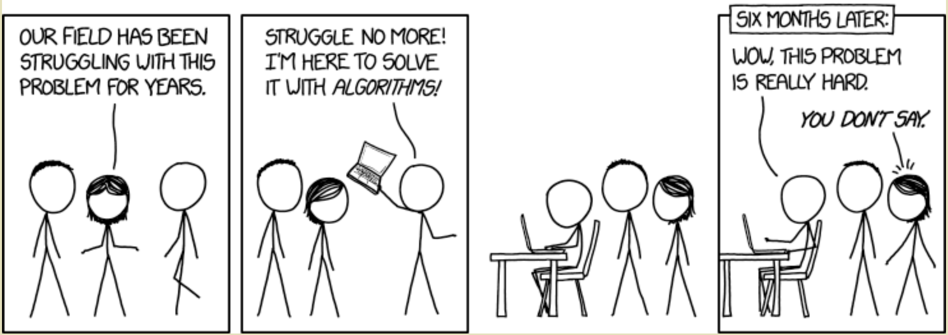
$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ with $([s_0 : s_1], [t_0 : t_1]) \mapsto [s_0 t_0 : s_0 t_1 : s_1 t_0 : s_1 t_1]$ gives



- The … ces along orien…
- Pro…
- Pro… for $① \circlearrowright$

Moduli spaces – semisimple case

Beyond these cases this gets very difficult



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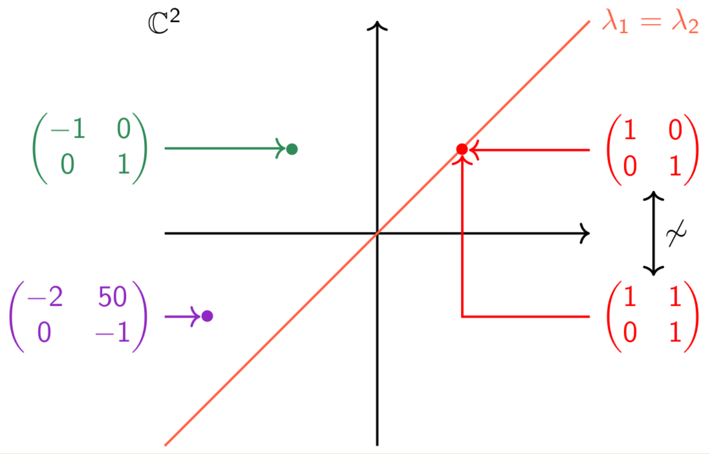
Note that this geometric approach is a bit better than the algebraic “generic” results



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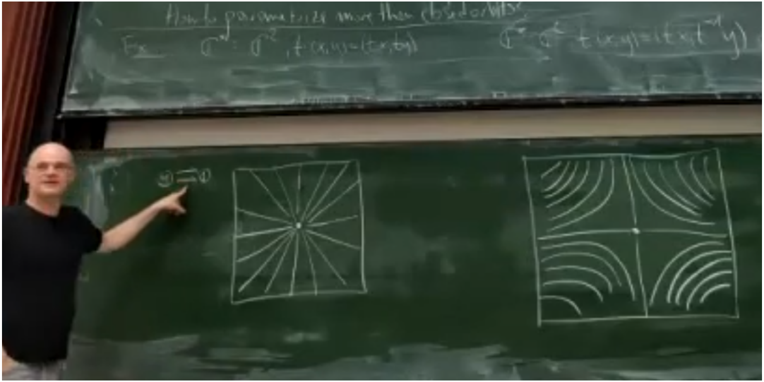

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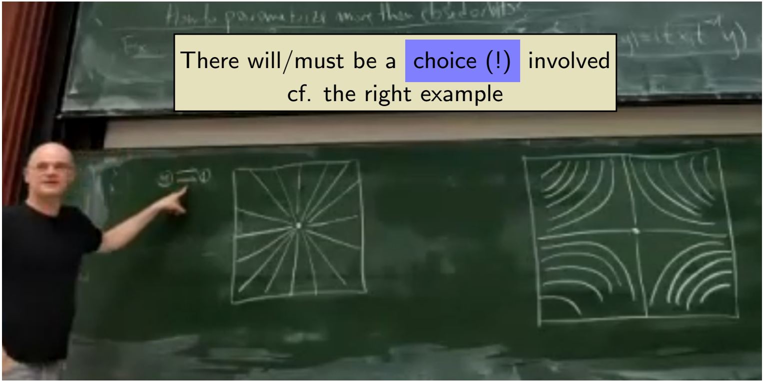

There will/must be a choice (!) involved
cf. the right example

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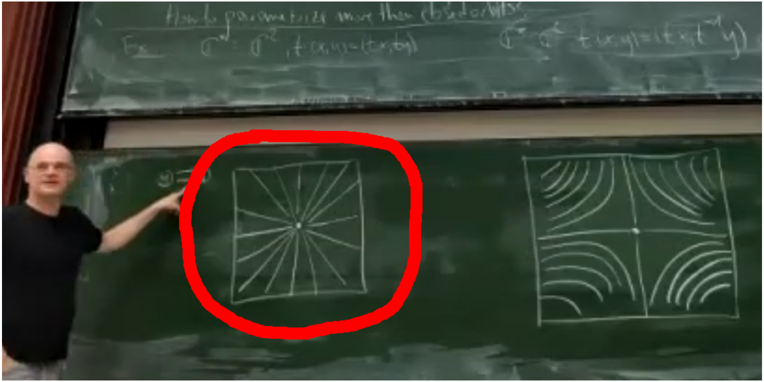

- Character $\chi = det$ (equals id since 1d case)
- χ-semistable points $R^{sst}_{(1,1)} \cong \mathbb{C}^2 \setminus \{(0,0)\}$
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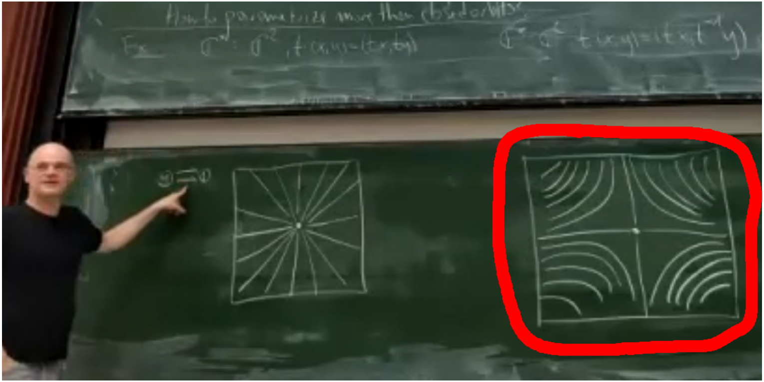

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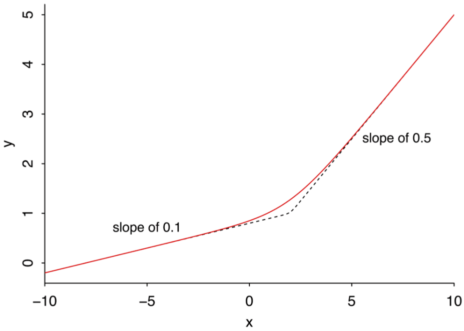

- Choose $\Theta \in (\mathbb{Z}\text{vertices})^*$, and define the slope $= \Theta\big(\boldsymbol{d}(V)\big)/\dim V \in \mathbb{Q}$
- Define

Θ-semistable	The slope is weakly decreasing on nontrivial(!) subreps
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Theorem (King, Schofield–van den Bergh ∼1994)

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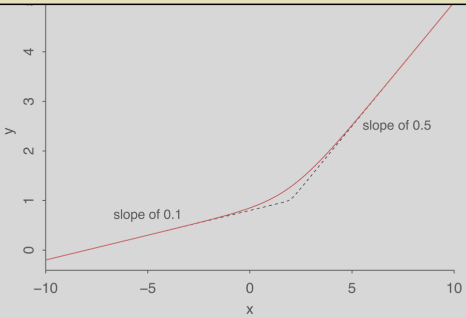

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Take $\Theta = 0$ so that $\chi(g) = 1$, then the slope is always zero

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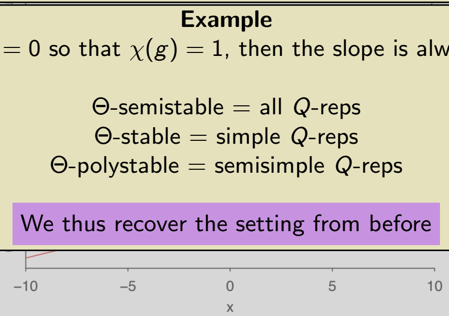

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To see this is nontrivial, but here is a sketch!
A Q-rep of dimension $(1, d)$ is a collection of m column vectors of size d
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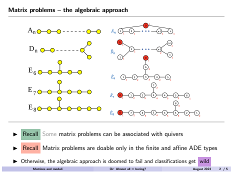
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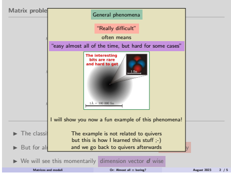
Matrices and moduli Or: Almost all = boring? August 2023 2 / 5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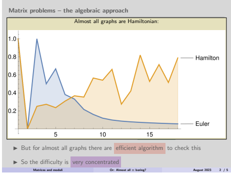
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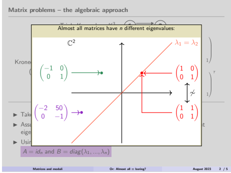
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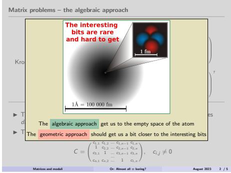
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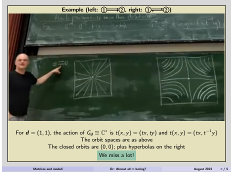
Matrices and moduli Or: Almost all = boring? August 2023 2 / 5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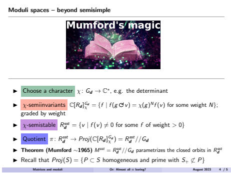
Matrices and moduli Or: Almost all = boring? August 2023 π / 5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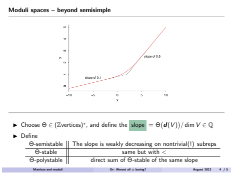
Matrices and moduli Or: Almost all = boring? August 2023 4 / 5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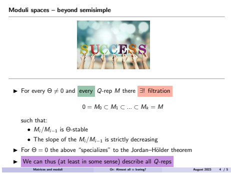
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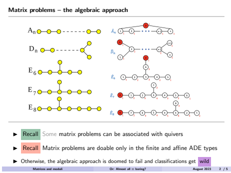
Matrices and moduli Or: Almost all = boring? August 2023 4 / 5

There is still much to do...



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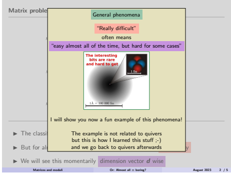

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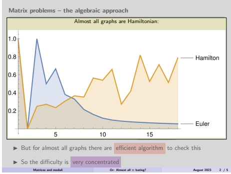

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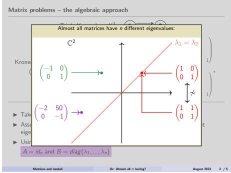

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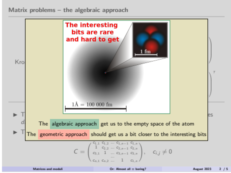

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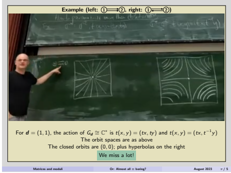

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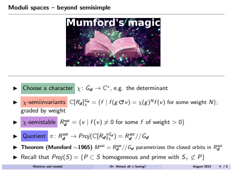

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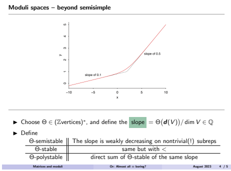
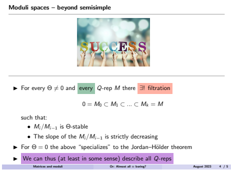
- **Theorem (Mumford ~1965)** $M^{sst} = R^{sst}_{\boldsymbol{d}}//G_{\boldsymbol{d}}$ parametrizes the closed orbits in $R^{sst}_{\boldsymbol{d}}$
- Recall that $Proj(S) = \{P \subset S \text{ homogeneous and prime with } S_+ \not\subset P\}$



Moduli spaces – beyond semisimple

- Choose $\Theta \in (\mathbb{Z}\text{vertices})^*$, and define the slope $= \Theta(\boldsymbol{d}(V))/\dim V \in \mathbb{Q}$
- Define

Θ-semistable	The slope is weakly decreasing on nontrivial(!) subreps
Θ-stable	same but with <
Θ-polystable	direct sum of Θ-stable of the same slope



Moduli spaces – beyond semisimple

- For every $\Theta \neq 0$ and every Q-rep M there $\exists!$ filtration

$$0 = M_0 \subset M_1 \subset ... \subset M_k = M$$

such that:
  - M_i/M_{i-1} is Θ-stable
  - The slope of the M_i/M_{i-1} is strictly decreasing
- For $\Theta = 0$ the above "specializes" to the Jordan–Hölder theorem
- We can thus (at least in some sense) describe all Q-reps

Thanks for your attention!