Matrices and quivers

Or: Complexity jumps

~~Accept~~Change what you cannot ~~change~~accept

August 2023

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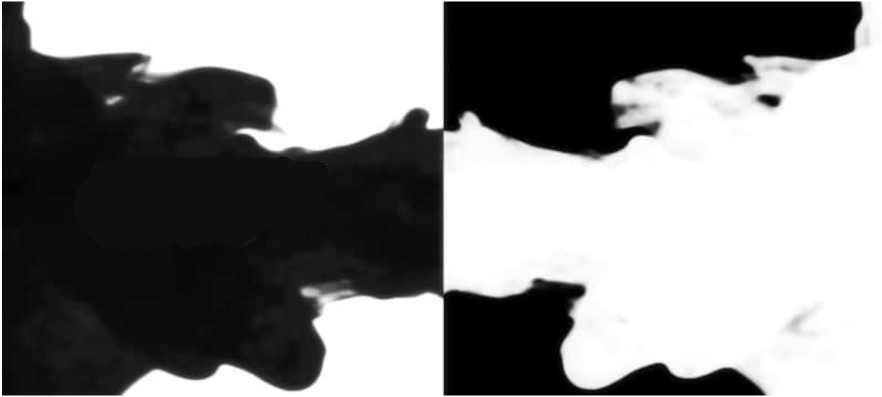
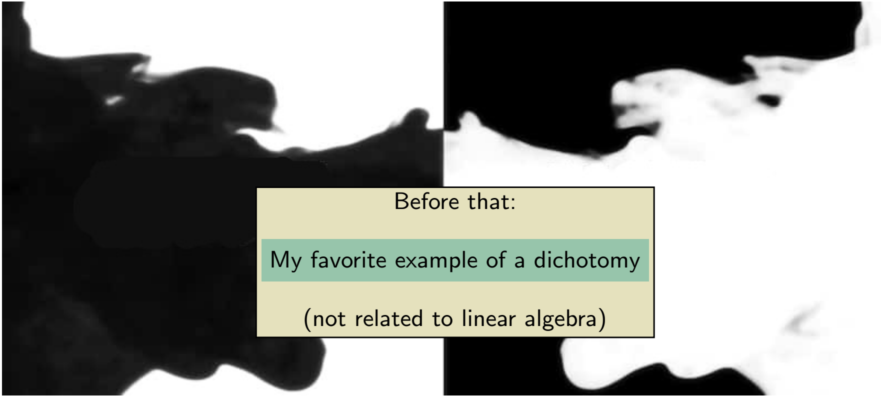
Matrix problems



- Dichotomy = division into two especially mutually exclusive or contradictory groups
- Slogan Dichotomy is everywhere
- Today My favorite linear algebra example of dichotomy

Matrix problems



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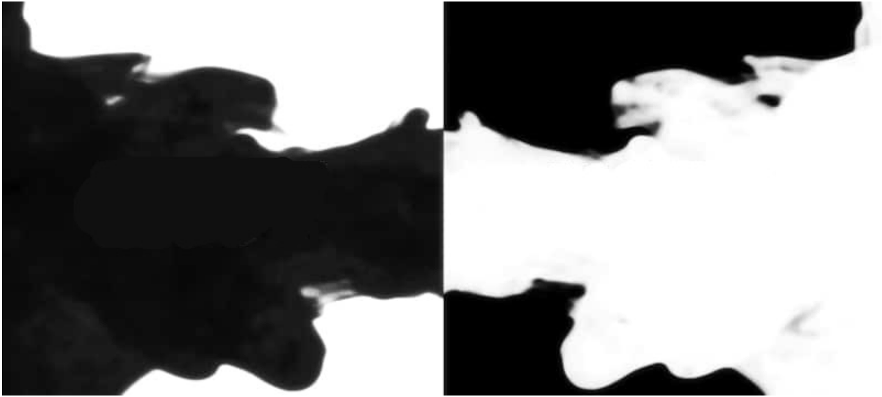
Matrix problems

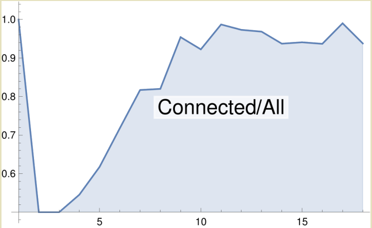


- **Metatheorem (0-1 theorem; folklore $\ll$1950)** Almost all properties of graphs are either false or true almost all of the time
- This works for almost all definitions of almost all
- Details are annoying, so let me rather give you two examples

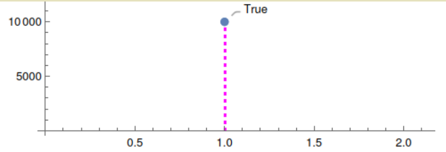
Example (of 1)

Theorem (folklore $\ll$1950) Almost all graphs are connected

Ratio connected/all:



10000 random graphs on 100 vertices:



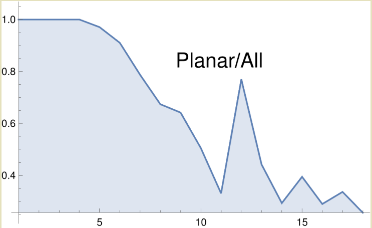

```
Counts[ListAut]
<|True → 10 000|>
```

▶ Details are annoying, so let me rather give you two examples

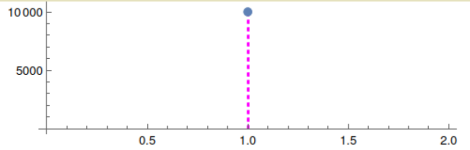
Example (of 0)

Theorem (folklore ≪1950) Almost no graph is planar

Ratio planar/all:



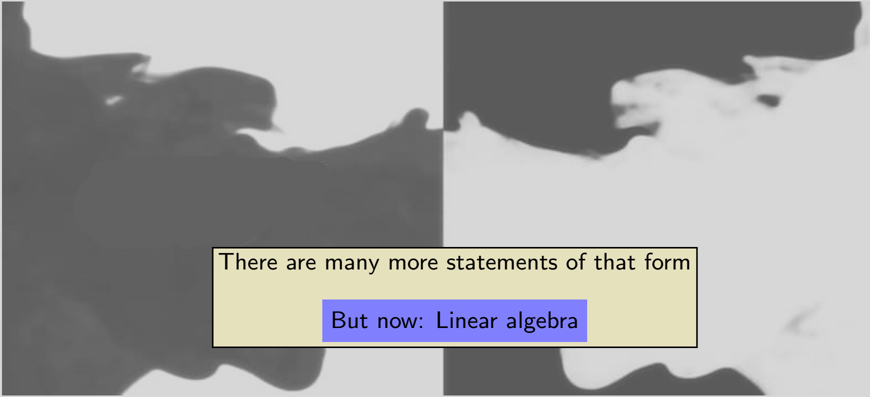
10000 random graphs on 100 vertices:




```
Counts[ListAut2]
<|False -> 10 000|>
```

- Details are annoying, so let me rather give you two examples

Matrix problems



There are many more statements of that form

But now: Linear algebra

- **Metatheorem (0-1 theorem; folklore ≪1950)** Almost all properties of graphs are either false or true almost all of the time
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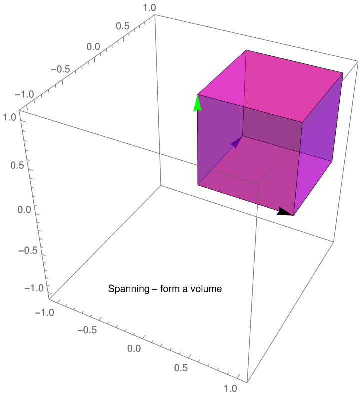
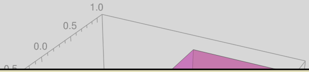
Matrix problems



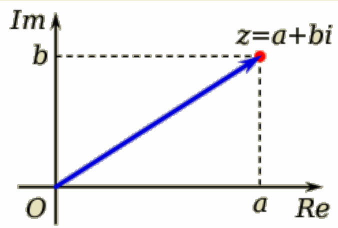
- Task Classify vector spaces up to isomorphism
- Solution **Theorem (folklore $\ll$1900)** The dimension determines the vector space
- Thus, vector spaces are classified by one discrete parameter

Matrix problems



So that this does not get completely bonkers:

I always work over the complex numbers and fdim. reps



That is however not always necessary

- Task Classify vector spaces up to isomorphism
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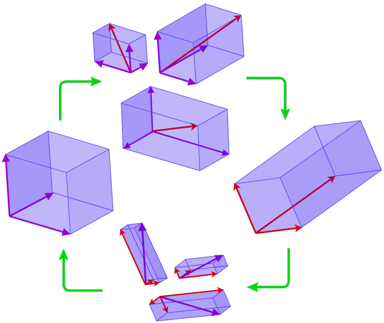
Matrix problems



- A natural equivalence relation on matrices is similarity:

$$(A \sim B) \Leftrightarrow (\exists P : A = P^{-1}BP)$$

  Similarity = A and B are the same linear automorphism up to base change

- Question How can we classify similar matrices?

Matrix problems

Jordan normal form (JNF):
$$\begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1\end{matrix}} & & & & \\ & \boxed{\begin{matrix}\lambda_2 & 1 \\ & \lambda_2\end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix}\lambda_n & 1 \\ & \lambda_n\end{matrix}} \end{pmatrix}$$

- **Theorem (Jordan ∼1870)** Two matrices are similar if and only if they have the same JNF
- Thus, similarity is classified by:

one discrete parameter $=$ size of the Jordan block

one continuous parameter $=$ eigenvalue of the Jordan block

Matrix problems

Jordan normal form (JNF): $\begin{pmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & 1 & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & 1 & & \\ & & & & \lambda_2 & & \\ & & & & & \lambda_3 & \\ & & & & & & \ddots \end{pmatrix}$

Vector space example For a fixed size, there is no continuous parameter

Jordan example For a fixed size, there is only one continuous parameter

Thus, there is at most one continuous parameter per fixed discrete parameter

- **Theorem (Jordan ∼1870)** Two matrices are similar if and only if they have the same JNF
- Thus, similarity is classified by:

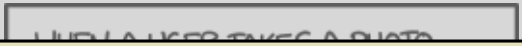
one discrete parameter = size of the Jordan block

one continuous parameter = eigenvalue of the Jordan block

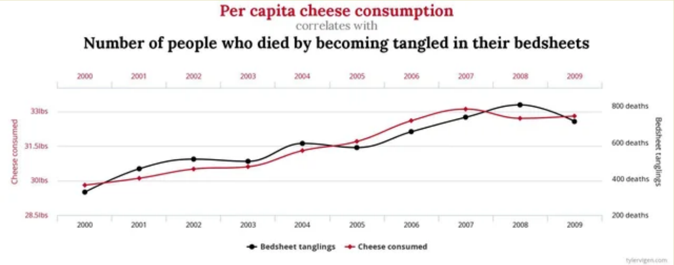
Matrix problems

- Similarity has a nice solution
- Simultaneous similarity $(A, B) \sim (P^{-1}AP, P^{-1}BP)$ (same P) is very difficult

Matrix problems



The problem is that A and B are unrelated
and making A "easier" doesn't imply anything about B



I will describe some approach to the
simultaneous similarity problem
but let us postpone that to the next talk

- Similarity has a
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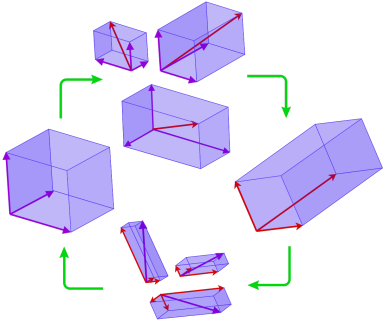
Matrix problems



$\sim \Leftrightarrow$ same linear auto. mod base change

$\approx \Leftrightarrow$ same linear map mod base change

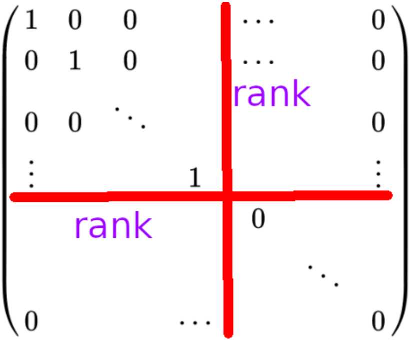
- Matrices $A = (A_1, ..., A_m)$ and $B = (B_1, ..., B_m)$ are simultaneously equivalent if:

$$(A \approx B) \Leftrightarrow (\exists P, Q : \forall i : A_i = Q^{-1} B_i P \text{ with } P, Q \text{ invertible })$$

Crucial: There is only one P and one Q

- Question How can we classify equivalent matrices?

Matrix problems



- **Theorem (folklore $\ll$1900)** Two matrices are equivalent if and only if they have the same nameless/Smith normal form as above
- Thus, equivalence for $m = 1$ is classified by:

one discrete parameter = the rank

Matrix problems

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}, \quad id_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$L_n = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{pmatrix}, \quad L_n^T = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{pmatrix}^T$$

- For $m = 2$ one has Kronecker's normal form (KNF) **Kronecker ∼1890**
- The KNF is similar to the JNF, but with four different blocks
- For $m = 2$ the classification is thus given by finitely many discrete parameters = sizes, types of blocks; and $\leq$ one continuous parameter = eigenvalue

Matrix problems

- Equivalence has a nice solution for $m = 1$ and is doable for $m = 2$
- For $m = 3$ this is extremely difficult

Observe complexity jumps:

Similarity $m = 0$ is trivial, $m = 1$ is ok, $m = 2$ is terrible

Equivalence $m = 1$ is easy, $m = 2$ is ok, $m = 3$ is terrible

- Equivale… $m = 2$
- For $m = 3$ this is extremely difficult

Matrix problems

SKETCH OF A MEMOIR ON ELIMINATION, TRANSFORMATION, AND CANONICAL FORMS.

1851

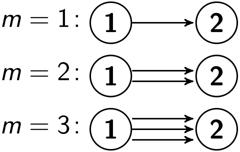
By J. J. SYLVESTER, M.A., F.R.S.

I now proceed to the consideration of the more peculiar branch of my inquiry, which is as to the mode of reducing Algebraical Functions to their simplest and most symmetrical, or as my admirable friend M. Hermite well proposes to call them, their Canonical forms. Every quadratic func-

Sylvester invented a great number of mathematical terms such as "matrix" (in 1850),[12] "graph" (in the sense of *network*)[13] and "discriminant".[14]

- Whenever there is a nice solution, then this was done quite a while ago $\ll$**1900**
- Next A different approach to these problems $\sim$**1950**

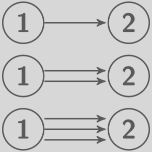
Quivers and matrices

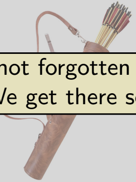


- The problem of simultaneous equivalence can be associated to a quiver
- Quiver = (finite) directed graph "It contains arrows"
- One then can formally prove that $m = 3$ is "impossible"

Quivers and matrices

$m = 1$:, $m = 2$:, $m = 3$:



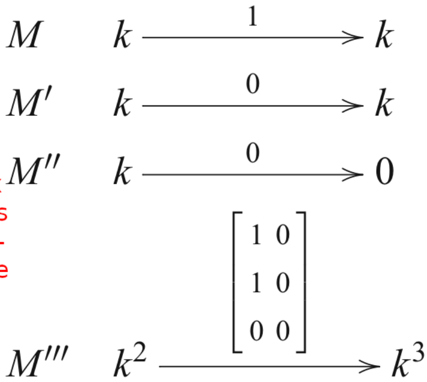


I have not forgotten similarity
We get there soon

- The problem of simultaneous equivalence can be associated to a quiver
- Quiver = (finite) directed graph "It contains arrows"
- One then can formally prove that $m = 3$ is "impossible"

Quivers and matrices

small k means complex numbers (the pictures are stolen)

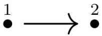
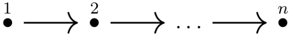
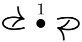


- A **representation of a quiver** ("a matrix problem for a quiver") is:
  - (i) A choice of a vector space for each vertex
  - (ii) A choice of a linear map for each edge

Why are these called representations?

Because every quiver Q has an associated algebra, P
the path algebra,
such that $Q\mathbf{Rep} \cong P\mathbf{Rep}$

Examples

	$\mathbf{k}[X]$
	$\begin{pmatrix} \mathbf{k} & \mathbf{k} \\ 0 & \mathbf{k} \end{pmatrix}$
	Upper triangular $n \times n$ matrices
	$\begin{pmatrix} \mathbf{k} & \mathbf{k} \oplus \mathbf{k} \\ 0 & \mathbf{k} \end{pmatrix}$
	$\mathbf{k}\langle X, Y\rangle$ (2 noncomm. free var's)

But we do not need to know that

- A rep
  - (i) A
  - (ii) A

Quivers and matrices

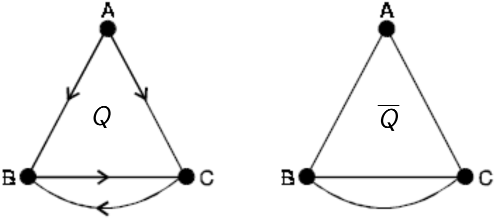
A matrix problem associated to a connected quiver Q without oriented cycles is...

(1) ...finite if and only if $\overline{Q}$ is of ADE type

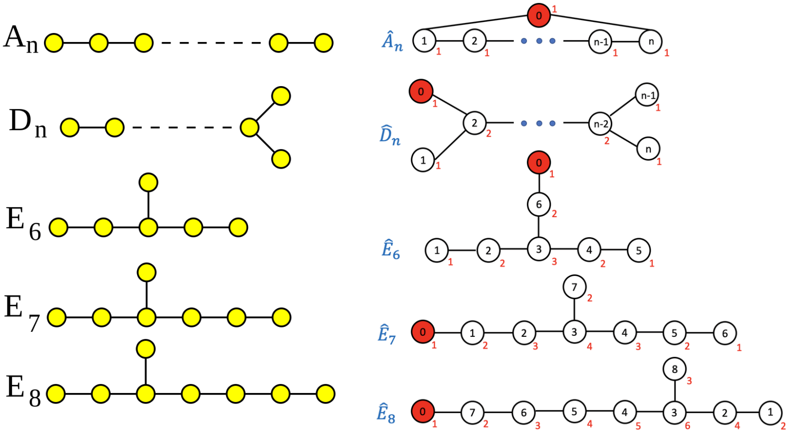
(2) ...infinite tame if and only if $\overline{Q}$ is of affine ADE type

(3) ...wild otherwise

- Finite = classification is given by finitely many discrete parameters; infinite tame = finitely many discrete and one continuous parameter; wild = forget it
- Q = the quiver; $\overline{Q}$ = the underlying graph



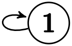
Quivers and matrices

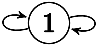


- ADE graphs and friends appear everywhere
- Left The ADE types; Right The affine ADE types

Quivers and matrices

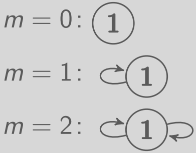
$m = 0$: 

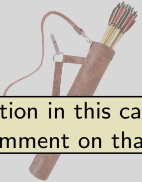
$m = 1$: 

$m = 2$: 

- The problem of simultaneous similarity can be associated to a quiver
- Quiver = (finite) directed graph "It contains arrows"
- One then can formally prove that $m = 2$ is "impossible"

Quivers and matrices





The classification in this case is not as nice
I comment on that later

- The problem of simultaneous similarity can be associated to a quiver
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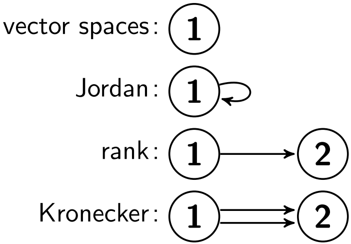
Quiver representations



- A representation of the vector space quiver is a choice of a vector space
- A representation of the Jordan quiver is a choice of a vector space and a linear map
- A representation of the rank quiver is a choice of two vector spaces and a linear map between them
- A representation of the Kronecker quiver is a choice of two vector spaces and two linear maps between them

Quiver representations

vector spaces: 1

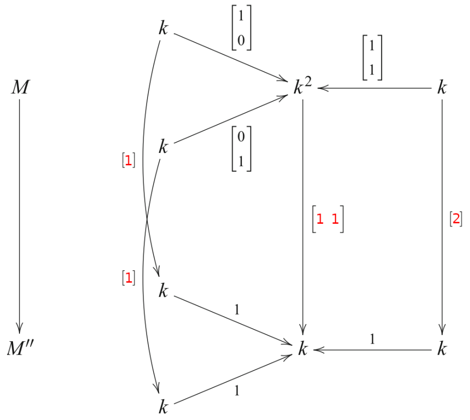
Goal

Design representations and equivalence of these representations such that the indecomposables mod iso correspond to the Jordan-type blocks

linear map between them

- A representation of the Kronecker quiver is a choice of two vector spaces and two linear maps between them

Quiver representations



- A morphism of quiver representations is a collection of linear maps satisfying the expected commuting diagram
- Equivalence is then defined with respect to isomorphism

Quiver representations

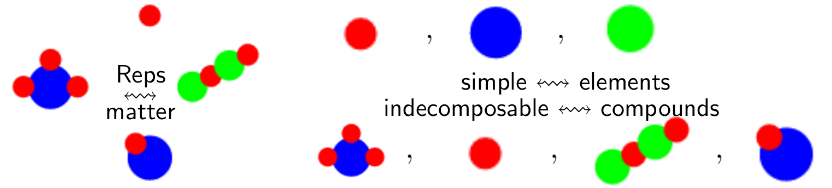
$$M \qquad k \xrightarrow{\;1\;} k \xleftarrow{\;0\;} 0;$$

$$M' \qquad k^2 \xrightarrow{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix}1\\ 1\end{bmatrix}} k.$$

Then the direct sum $M \oplus M'$ is the representation

$$k \oplus k^2 \xrightarrow{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}} k \oplus k^2 \xleftarrow{\begin{bmatrix}0 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}} 0 \oplus k;$$

- **Lemma/Fact** Quiver representations form a Krull–Schmidt abelian category so the usual Yoga works
- Goal Classify simple and/or indecomposable representations



Picture stolen from Geordie Williamson

Simple = no substructure, indecomposable = $M \cong X \oplus Y$ implies $X \cong 0$ or $Y \cong 0$

These are very different!

$$k \oplus k^2 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} k \oplus k^2 \xleftarrow{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}} 0 \oplus k;$$

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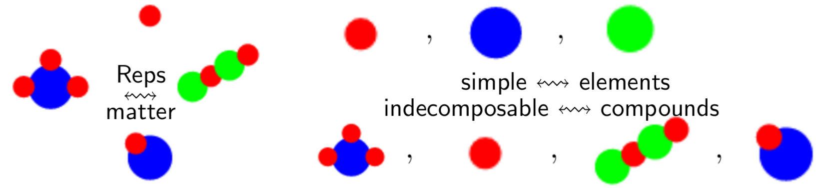

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Semisimple ⇔ simple=indecomposable ⇔ the quiver has no edges Semisimplicity is rare

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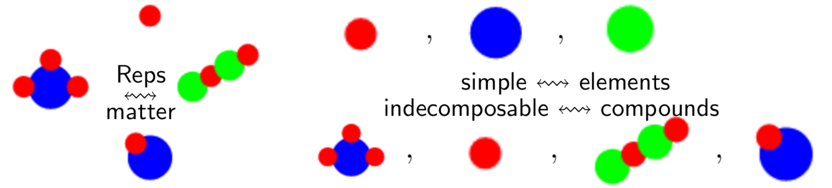

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Simple = no substructure, indecomposable = $M \cong X \oplus Y$ implies $X \cong 0$ or $Y \cong 0$
These are very different!

Semisimple ⇔ simple=indecomposable ⇔ the quiver has no edges **Semisimplicity is rare**

Example

The Jordan quiver has a one parameter family of 1d simples (up to iso – I drop this)
But arbitrary dim. indecomposables ⇝ Jordan blocks

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

Quiver representations

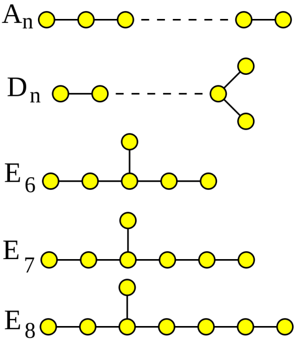
$$S(3) \cong \begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longleftarrow & k & \longleftarrow & 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array},$$

$$P(3) \cong \begin{array}{ccccccc} 0 & \longrightarrow & k & \xleftarrow{1} & k & \longleftarrow & 0 \\ & & & & \downarrow{\scriptstyle 1} & & \\ & & & & k & & \end{array},$$

$$I(3) \cong \begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longleftarrow & k & \xleftarrow{1} & k \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}.$$

- **Lemma/Fact** For any fdim algebra A $\exists$ a quiver Q and an exact functor $A\mathbf{Rep} \to Q\mathbf{Rep}$ preserving inde.
- The point Quiver representations are really easy
- **Example (fdim case)** Simples $\leftrightsquigarrow$ one vertex, inde. projective $\leftrightsquigarrow$ outgoing, inde. injective $\leftrightsquigarrow$ incoming; # simple/inde. proj./inde. inj. $=$ # vertices

Quiver representations



- **Theorem (Yoshii ∼1956, Gabriel ∼ 1972)** A connected quiver Q without oriented cycles has finitely many indecomposables if and only if $\overline{Q}$ is of ADE type
- In this case # indecomposables = # positive roots Discrete parameters!

Quiver representations

Example (rank quiver)

$$S(2) \qquad (0 \longrightarrow k),$$

$$M \qquad (k \xrightarrow{1} k),$$

$$S(1) \qquad (k \longrightarrow 0).$$

The rank quiver has three indecomposables
M corresponds to the rank parameter

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$$M \qquad (k \xrightarrow{\;1\;} k),$$

$$S(1) \qquad (k \longrightarrow 0).$$

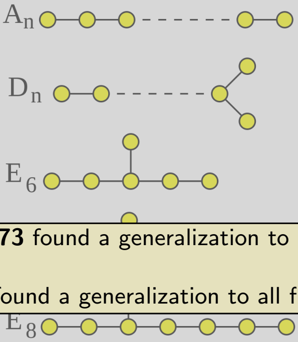
The rank quiver has three indecomposables
M corresponds to the rank parameter

Example (type A)

Indecomposables can be identified with consecutive strings of $0 = 0$ and $1 = \mathbb{C}$
e.g. 100, 010, 001, 110, 011 and 111

- out ype
- In this case # indecomposables = # positive roots Discrete parameters!

Quiver representations

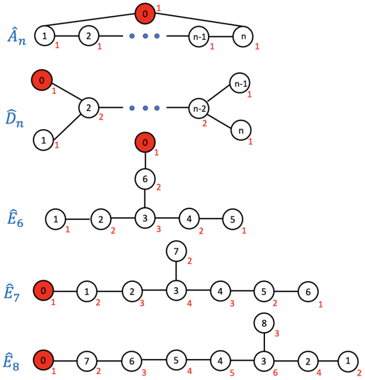


Dlab–Ringel ∼1973 found a generalization to all finite Dynkin types

Heng ∼2023 found a generalization to all finite Coxeter types

- **Theorem (Yoshii ∼1956, Gabriel ∼ 1972)** A connected quiver Q without oriented cycles has finitely many indecomposables if and only if $\overline{Q}$ is of ADE type
- In this case # indecomposables = # positive roots Discrete parameters!

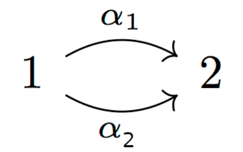
Quiver representations



- **Theorem (Donovan–Freislich, Nazarova ∼1973)** A (usual adjectives) quiver Q has tame rep type if and only if $\overline{Q}$ is of finite or affine ADE type
- Tame = indecomposables can form countably many one-parameter families; infinite tame = tame but not finite

Quiver re

Example (Kronecker quiver)



Indecomposables of the Kronecker quiver $\leftrightsquigarrow$

Class 1 $\mathbb{C}^n \rightrightarrows \mathbb{C}^{n+1}$ with $(id_n, 0)$ and L_n

Class 2 $\mathbb{C}^{n+1} \rightrightarrows \mathbb{C}^n$ with $(id_n, 0)^T$ and L_n^T

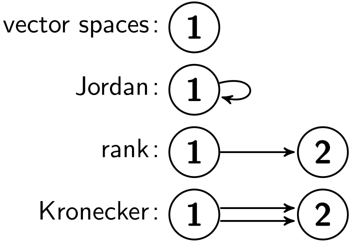
Class 3 $\mathbb{C}^n \rightrightarrows \mathbb{C}^n$ with id_n and $J_n(\lambda)$

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}, \quad id_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$L_n = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{pmatrix}, \quad L_n^T = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \end{pmatrix}^T$$

- **Theo** ... tives)
  quive ... DE type
- Tame ... families;
  infinit

Quiver representations



- **ADE Theorem** $\Rightarrow$ the vector space quiver has inde. given by $\mathbb{C}$
- The Jordan quiver has inde. given by Jordan blocks
- **ADE Theorem** $\Rightarrow$ the rank quiver has inde. given by $\mathbb{C} \to 0$ with zero, $0 \to \mathbb{C}$ with zero and $\mathbb{C} \to \mathbb{C}$ with identity
- **Affine ADE Theorem** $\Rightarrow$ the Kronecker quiver has inde. given as before

Quiver representations

vector spaces: 1

To be more precise:
one still needs to work to get the actual
classification; the theorems only
give an overall parametrization scheme

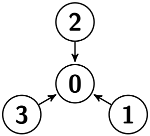
$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}, \quad id_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$L_n = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 & 1 \end{pmatrix}, \quad L_n^T = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 & 1 \end{pmatrix}^T$$

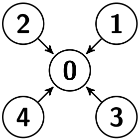
- ADE
- The
- ADE ... zero, $0 \to$ ...
- Affin ... s before

Quiver representations

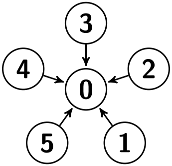
3 subspaces:



4 subspaces:



5 subspaces:



- **Subspace problem** Classify $V_1, ..., V_m \subset V_0$ up to $(V_1, ..., V_m) \equiv (W_1, ..., W_m)$ if $\exists$ iso. $f \colon V_0 \to V_0$ with $f(V_i) = W_i$
- Above $m = 3, 4, 5$ as quiver problems

We get:

3 subspace problem Discrete

The 3-subspace problem is of finite representation type (D_4); the indecomposables are (up to "permutation of legs"):

$$\begin{array}{ccccc} 0 & \xrightarrow{0} & \mathbf{k} & \xleftarrow{0} & 0 \\ & & {\scriptstyle 0}\uparrow & & \\ & & 0 & & \end{array} \quad \begin{array}{ccccc} \mathbf{k} & \xrightarrow{0} & 0 & \xleftarrow{0} & 0 \\ & & {\scriptstyle 0}\uparrow & & \\ & & 0 & & \end{array} \quad \begin{array}{ccccc} \mathbf{k} & \xrightarrow{1} & \mathbf{k} & \xleftarrow{0} & 0 \\ & & {\scriptstyle 0}\uparrow & & \\ & & 0 & & \end{array}$$

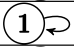
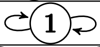
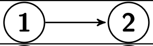
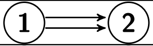
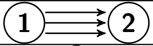
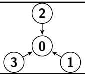
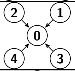
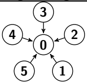
$$\begin{array}{ccccc} \mathbf{k} & \xrightarrow{1} & \mathbf{k} & \xleftarrow{0} & 0 \\ & & {\scriptstyle 1}\uparrow & & \\ & & \mathbf{k} & & \end{array} \quad \begin{array}{ccccc} \mathbf{k} & \xrightarrow{\binom{1}{0}} & \mathbf{k}\oplus\mathbf{k} & \xleftarrow{\binom{0}{1}} & \mathbf{k} \\ & & {\scriptstyle \binom{1}{1}}\uparrow & & \\ & & \mathbf{k} & & \end{array}$$

4 subspace problem One parameter

5 subspace problem Wild (for now: wild=¬tame)

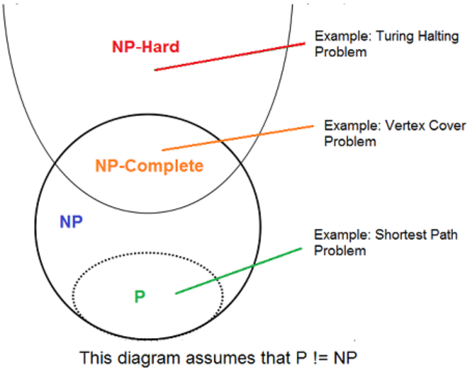
- S
$(V_1, \dots, V_m) \cong (W_1, \dots, W_m)$ if $\exists$ iso. $f : V_0 \to V_0$ with $f(V_i) = W_i$
- Above $m = 3, 4, 5$ as quiver problems

Quiver representations

Problem	Classification	Quiver
Vector space	Discrete	
Equivalence	One parameter	
Double equivalence	Wild	
Similarity	Discrete	
Double similarity	One parameter	
Triple similarity	Wild	
3 subspace	Discrete	
4 subspace	One parameter	
5 subspace	Wild	

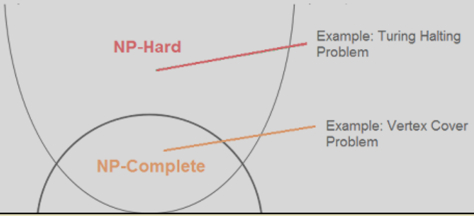
- Mind the gap!

Dichotomy (or trichotomy, depending on who you ask)



- Q has wild representation type if, for each fdim algebra A, there exists an exact functor $\mathcal{I}\colon A\mathbf{Rep} \to Q\mathbf{Rep}$ preserving inde. Similar to NP complete
- Classifying inde. Q-reps for wild Q implies that we can do the same for any finite dimensional algebra

Dichotomy (or trichotomy, depending on who you ask)



Tame ⇝ at most one-parameter families of inde.

Wild ⇝ at least as complicated as any other quiver

We will use the same notions for finite dimensional algebras

- Q has wild representation type if, for each fdim algebra A, there exists an exact functor $\mathcal{I}\colon A\mathbf{Rep} \to Q\mathbf{Rep}$ preserving inde. Similar to NP complete
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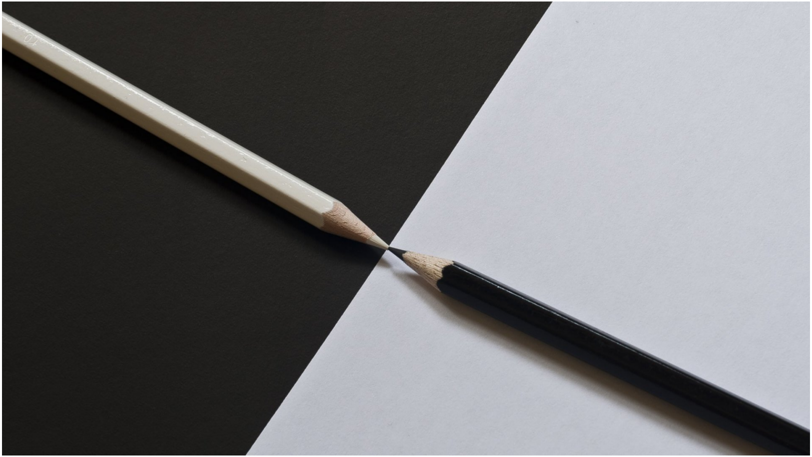
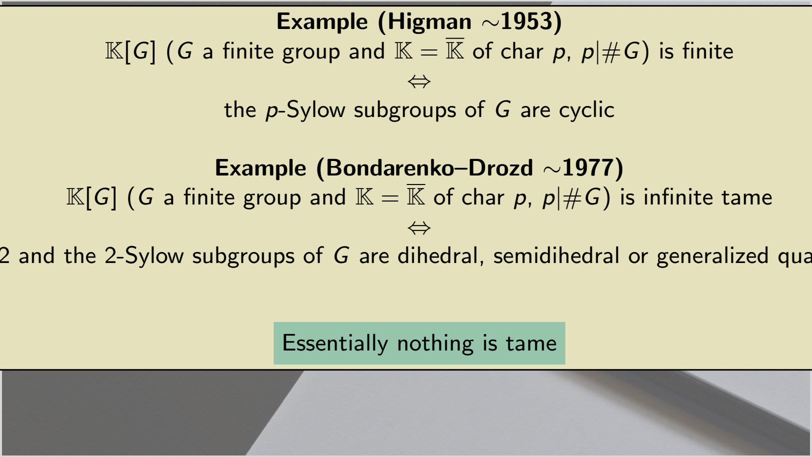


- **Theorem (Drozd $\sim$1977)** A quiver is either tame or wild
- **Theorem (Drozd $\sim$1977)** A finite dimensional algebra is either tame or wild



Example (Higman ∼1953)

$\mathbb{K}[G]$ (G a finite group and $\mathbb{K}=\overline{\mathbb{K}}$ of char p, $p|\#G$) is finite

$\Leftrightarrow$

the p-Sylow subgroups of G are cyclic

Example (Bondarenko–Drozd ∼1977)

$\mathbb{K}[G]$ (G a finite group and $\mathbb{K}=\overline{\mathbb{K}}$ of char p, $p|\#G$) is infinite tame

$\Leftrightarrow$

$p=2$ and the 2-Sylow subgroups of G are dihedral, semidihedral or generalized quaternion

Essentially nothing is tame

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- **Theorem (Drozd ∼1977)** A finite dimensional algebra is either tame or wild

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Essentially nothing is tame

Example

The symmetric group $S_n = \mathrm{Aut}(\{1, ..., n\})$ is finite/$\mathbb{C}$

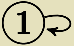
Example (Putcha ∼1997)

The transformation monoid $T_n = \mathrm{End}(\{1, ..., n\})$ is wild/$\mathbb{C}$ unless $n \leq 4$

- T
- **Theorem (Drozd ∼1977)** A finite dimensional algebra is either tame or wild

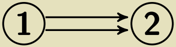
The infinite setting gets much more difficult (and that is why its skipped)

Example



is tame and TAME

Example (Ringel ∼1979)



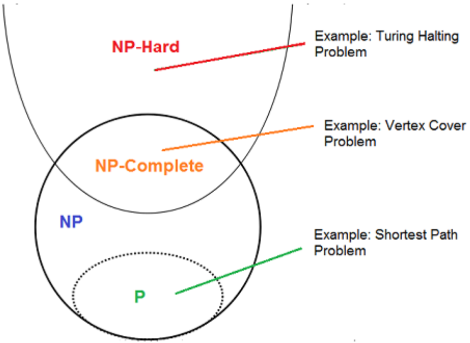
is tame but WILD

Capital spelling = same as before but including ∞-dim. reps

Classifying inde. of the Jordan quiver
is the same as classifying them for $\mathbb{C}[X]$
and $\mathbb{C}[X]$ is a PID so $M \cong (\text{free})_i \oplus (\text{fdim})_j$

- **Theorem (Drozd ∼1977)** A quiver is either tame or wild
- **Theorem (Drozd ∼1977)** A finite dimensional algebra is either tame or wild

Dichotomy (or trichotomy, depending on who you ask)



This diagram assumes that P != NP

- As above, this is not the end of the line
- **Theorem (Belitskii–Sergeichuk ∼2007)** Classifying trilinear forms contains the problem of classifying inde. of any finite dimensional algebra, but not vice versa More like NP-hard

Dichotomy (or trichotomy, depending on who you ask)

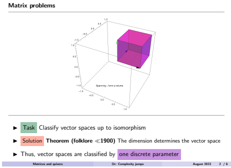
bilinear: cap = ⌒ : $\mathtt{X} \otimes \mathtt{X} \to \mathbb{1}$, trilinear: tup = ⏉ : $\mathtt{X} \otimes \mathtt{X} \otimes \mathtt{X} \to \mathbb{1}$

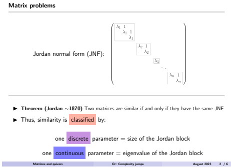
Theorem (Horn–Sergeichuk ∼2006, but parts are much older)
Classification of bilinear forms ⇔ classification of matrix congruence $A = P^T BP$ with normal form pieces given by $J_n(0)$ and

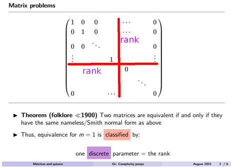
$$\mathsf{G}_n = \begin{pmatrix} & & & (-1)^{n+1} & (-1)^{n} \\ & & \iddots & \iddots & \\ & -1 & -1 & & \\ 1 & 1 & & & \end{pmatrix}, \quad \mathsf{H}_{2n}(\lambda) = \left(\begin{array}{c|c} 0 & id_n \\ \hline \mathsf{J}_n(\lambda) & 0 \end{array}\right)$$

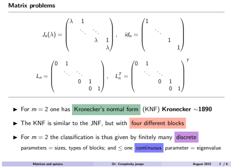
The trilinear analog is beyond hopeless

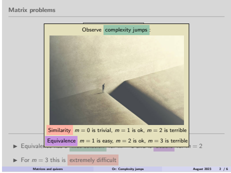
- As above, this is not the end of the line
- **Theorem (Belitskii–Sergeichuk ∼2007)** Classifying trilinear forms contains the problem of classifying inde. of any finite dimensional algebra, but not vice versa More like NP-hard

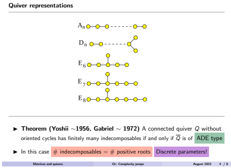


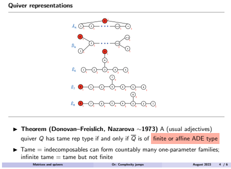


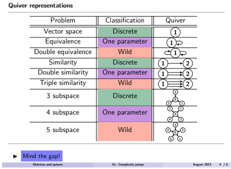


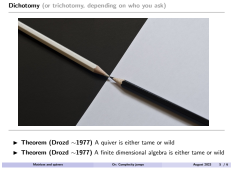




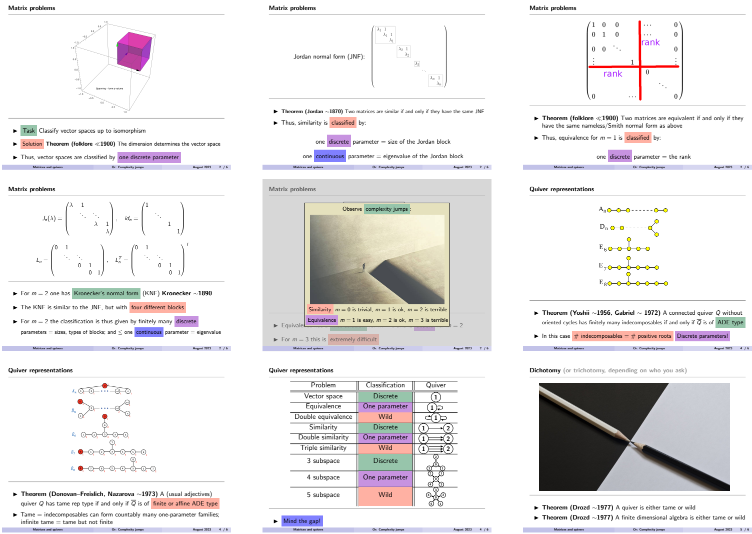








There is still much to do...



Thanks for your attention!