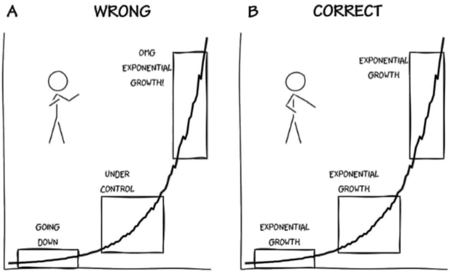
# Growth rates in tensor powers

Or: Jupiter and friends

~~Accept~~Change what you cannot ~~change~~accept




I report on work of Kevin Coulembier, Pavel Etingof and Victor Ostrik

July 2023

## Let us not count!


- $\Gamma$ = any affine semigroup superscheme, $\mathbb{K}$ = any ground field, $V$ = any fin dim $\Gamma$-rep
- $\Gamma$ has the notion of a tensor product
- Problem Decompose $V^{\otimes n}$; note that $\dim V^{\otimes n} = (\dim V)^n$

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dim $V = 1$ works perfectly well
but then my story about exponential growth is flawed
so I ignore dim $V = 1$ and assume dim $V > 1$

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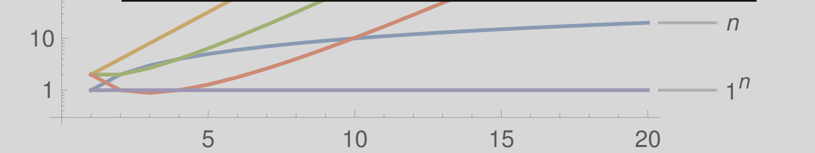
We go by examples!

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We go by examples!

**Examples**

Any finite group, monoid, semigroup

Symmetric groups, alternating groups, cyclic groups, the monster, $GL_N(\mathbb{F}_{p^k})$, ...

Actually any group, monoid, semigroup

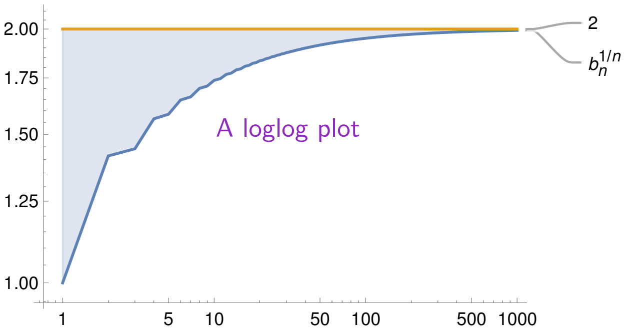
$GL_N(\mathbb{C})$, $GL_N(\mathbb{R})$, $GL_N(\overline{\mathbb{F}_{p^k}})$, symplectic, orthogonal, braid groups, Thompson groups, ...

Super versions

$GL_{M|N}$, $OSP_{M|2N}$, periplectic, queer, ...

Slogan This is a very general setting

## Let us not count!




- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $2 = \dim V$: $\sqrt[1000]{b_{1000}} \approx 1.99265$

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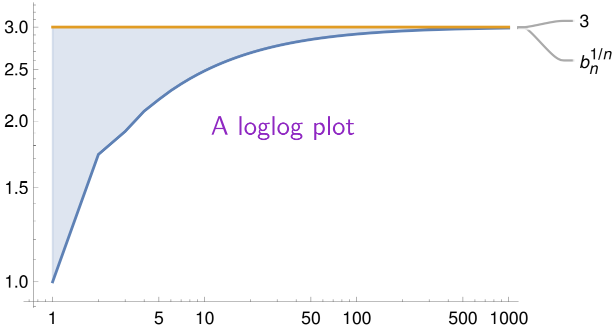


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$$\{1,1,3,7,19,51,141,393,1107,3139,8953\}, \quad b_n \text{ for } n = 0,...,10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $3 = \dim V$: $\sqrt[1000]{b_{1000}} \approx 2.9875$

## Let

### Observation 1

Whatever is true for $SL_2$ over $\mathbb{C}$ is true in general, right?

So let us come back to the general setting:
$\Gamma$ = affine semigroup superscheme
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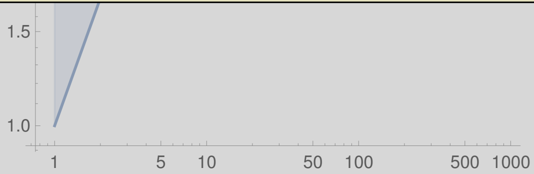


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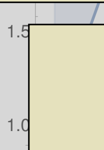


**Observation 2**

$b_n b_m \leq b_{n+m} \Rightarrow$
$\beta = \lim_{n\to\infty} \sqrt[n]{b_n}$
is well-defined by a version of Fekete's Subadditive Lemma

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1.5

1.0

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### Observation 3

$$1 \leq \beta \leq \dim V$$

$\beta = 1 \Leftrightarrow V^{\otimes n}$ for $n \gg 0$ is 'one block'

$\beta = \dim V \Leftrightarrow$ summands of $V^{\otimes n}$ for $n \gg 0$ are 'essentially one-dimensional'

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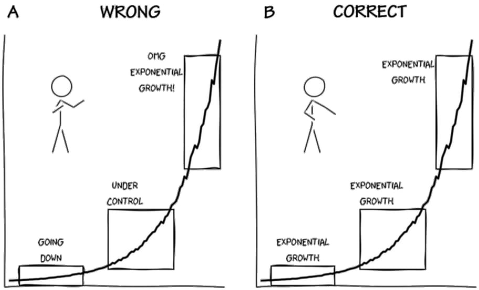
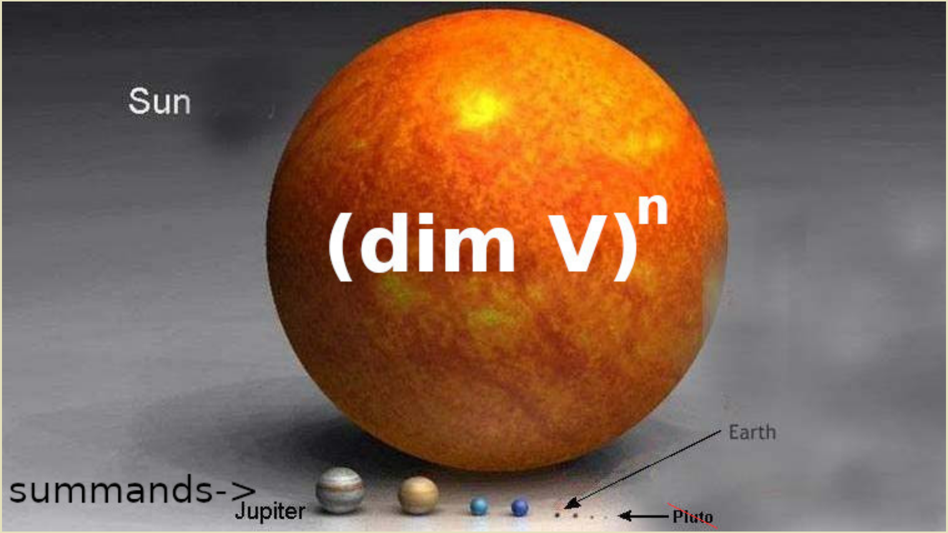



We have

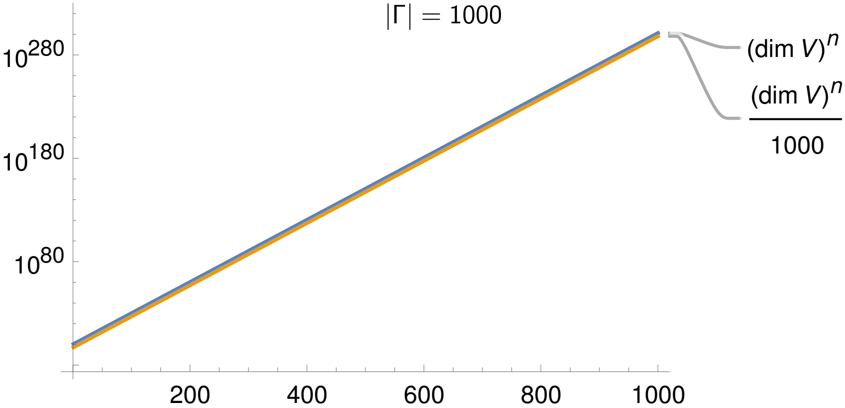
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim V$$

## Exponential growth is scary

In other words, compared to the size of the exponential growth of $(\dim V)^n$ all indecomposable summands are 'essentially one-dimensional'

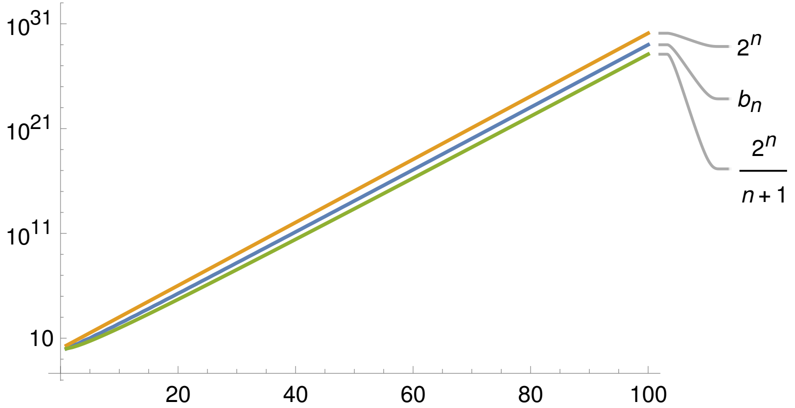


## The semisimple case and Jupiter



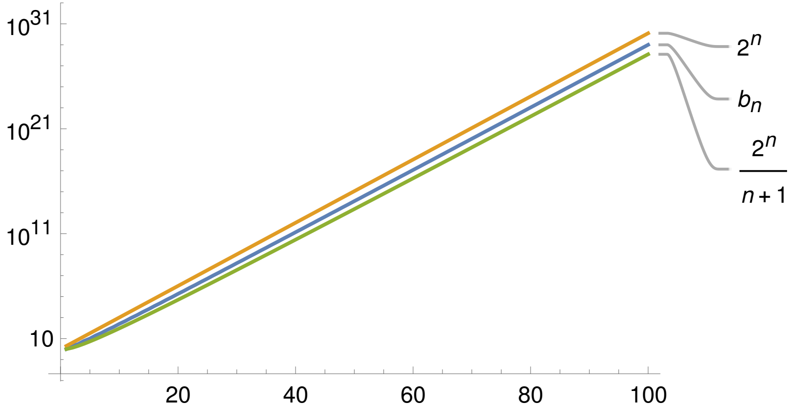

- $\Gamma$ = a finite group of order $|\Gamma| = 1000$, $\mathbb{K} = \mathbb{C}$
- Every indecomposable $\Gamma$-rep $Z$ has $\dim Z \leq |\Gamma| = 1000$
- Assume every $Z$ is Jupiter $\Rightarrow$ $(\dim V)^n/1000 \leq b_n \leq (\dim V)^n$ $\Rightarrow$ Done!

## The semisimple case and Jupiter




- $\Gamma = SL_2$ with $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$
- Every indecomposable $G$-rep $Z$ in $V^{\otimes n}$ has $\dim Z \leq n+1$ (top is $Sym^n\mathbb{C}^2$)
- Assume every $Z$ is Jupiter $\Rightarrow$ $(\dim V)^n/(n+1) \leq b_n \leq (\dim V)^n$ $\Rightarrow$ Done!

## The semisimple case and Jupiter




Ditto for any other $V = Sym^k\mathbb{C}^2$

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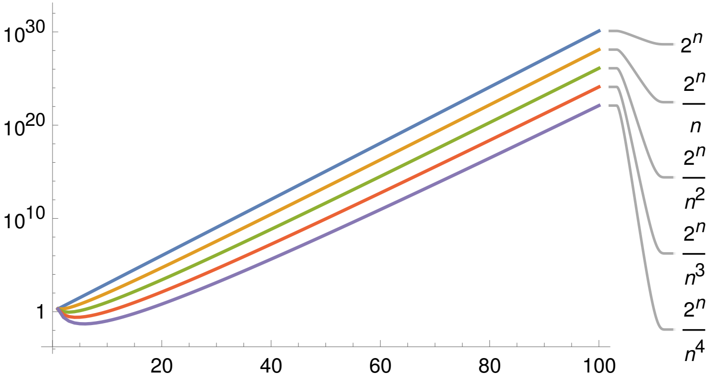
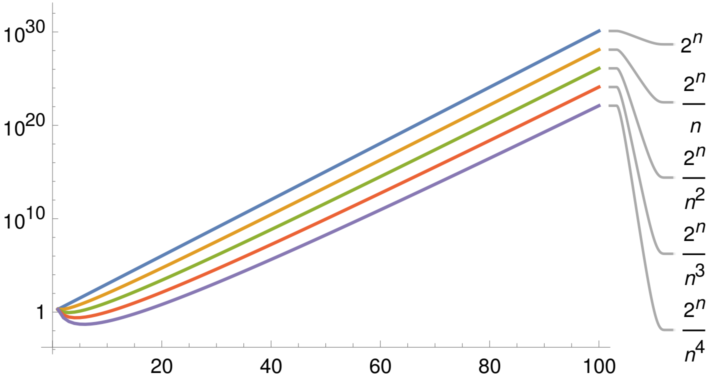



- $\Gamma = SL_M$ with $\mathbb{K} = \mathbb{C}$, $V =$ any fin dim $\Gamma$-rep
- Every indecomposable $G$-rep $Z$ in $V^{\otimes n}$ has $\dim Z \leq$ some poly in weights (Weyl's dim formula, e.g. $\dim V_{m_1,m_2} = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$)
- Assume every $Z$ is Jupiter $\Rightarrow$ $(\dim V)^n/$some poly in weights $\leq b_n \leq (\dim V)^n$ $\Rightarrow$ Done!

## The semisimple case and Jupiter




Ditto for $\Gamma = GL_M$

- $\Gamma = SL_M$ with $\mathbb{K} = \mathbb{C}$, $V =$ any fin dim $\Gamma$-rep
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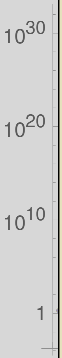
# The semisimple case and Jupiter




**The embedding trick**

**Case 1** $\Gamma$ is an affine group scheme

$\Rightarrow \Gamma \hookrightarrow GL(V)$

$\Rightarrow b_n^{GL(V),V} \leq b_n^{\Gamma,V}$

$\Rightarrow$ Done

(we just had $GL(V)$)

**Case 2** $\Gamma$ is an affine semigroup scheme

$\Rightarrow \Gamma \hookrightarrow END(V)$

$\Rightarrow (b_n^{END(V),V} \leq b_n^{\Gamma,V})$ + use $(b_n^{GL(V),V} = b_n^{END(V),V})$ (omitted)

$\Rightarrow$ Done

**Case 3** $\Gamma$ is something super

$\Rightarrow$ use the same trick but for $GL_{M|N}$ Now live

$\Rightarrow$ Done

- $\Gamma = SL$
- Every indecomposable $G$-rep $Z$ in $V^{\otimes n}$ has $\dim Z \leq$ some poly in weights (Weyl's dim formula, e.g. $\dim V_{m_1,m_2} = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$)
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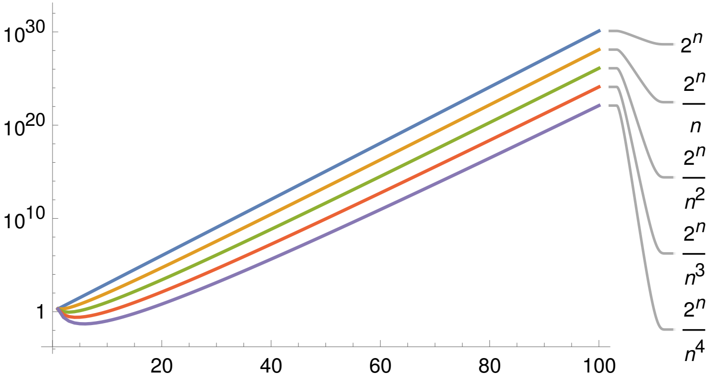



- $\Gamma = GL_{M|N}$ with $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^{M|N}$
- Every indecomposable $G$-rep $Z$ in $V^{\otimes n}$ has $\dim Z \leq$ some poly in weights (**Theorem (Berele–Regev ∼1987)** $V^{\otimes n}$ is semisimple!)
- Assume every $Z$ is Jupiter $\Rightarrow$ $(\dim V)^n /$some poly in weights $\leq b_n \leq (\dim V)^n$ $\Rightarrow$ Done!

## The sem

$10^3$

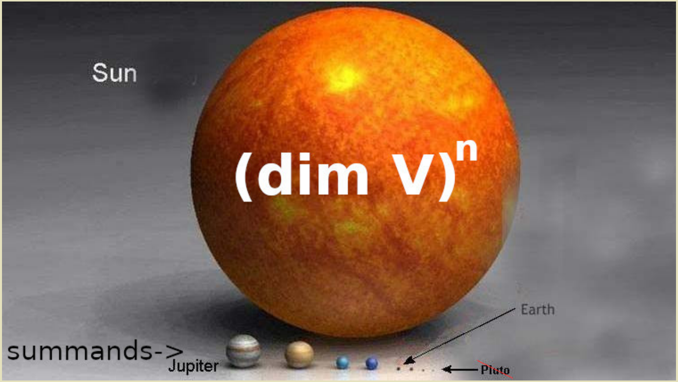
$10^2$

$10^1$

### Summary (semisimple case)

We ‘know’ the characters and dimensions of the indecomposables
They do not grow fast enough to compete with exponential growth
Dividing by Jupiter (= worst case) proves

$$\beta = \dim V$$




- $\Gamma =$
- Every ... eights
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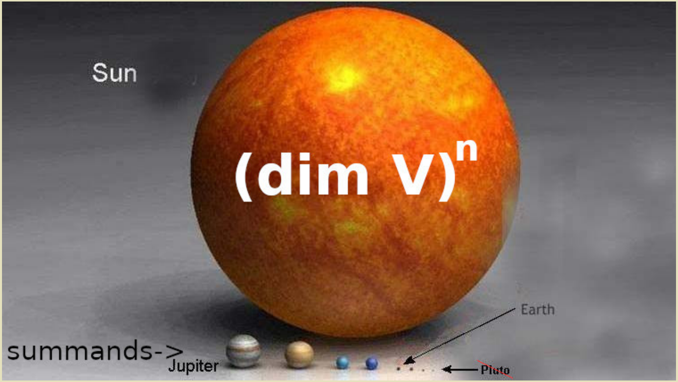
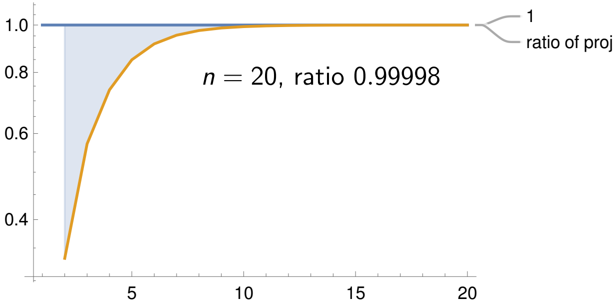



- $\Gamma =$
- Every ... eights
  (Theo...

Turns out that the nonsemisimple case is not much different

- Assume every Z is Jupiter $\Rightarrow$ $(\dim V)^n$/some poly in weights $\leq b_n \leq (\dim V)^n$ $\Rightarrow$ Done!

## The nonsemisimple case and Jupiter



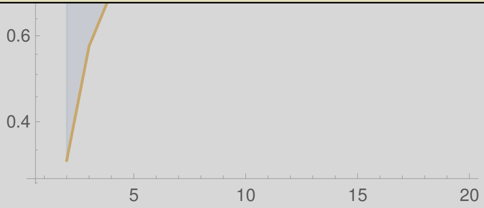

- $\Gamma$ = a finite group
- **Theorem (Bryant–Kovács ∼1972)** Essentially all summands of $V^{\otimes n}$ are 'projective' The projective cone
- Every indecomposable projective $\Gamma$-rep $P$ has dim $P \leq |\Gamma|$
- Non-projective summands 'do not matter' and play the Jupiter argument

The projective cone

"projective $\otimes$ anything $=$ projective"

Bryant–Kovács: eventually there will be some projective in $V^{\otimes k}$ (say for $V$ faithful)

$\Rightarrow$ essentially all summands of $V^{\otimes n}$ are projective for $n \gg 0$

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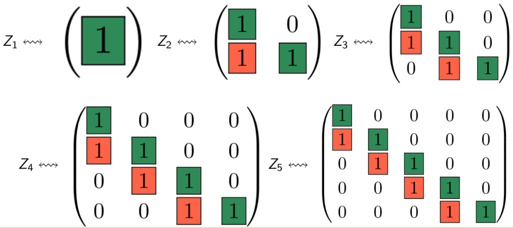
**Example (the picture you just saw)**

$\Gamma = \mathbb{Z}/5\mathbb{Z}$, $\mathbb{K} = \overline{\mathbb{F}_5}$, $V = Z_3 =$ 3d indecomposable, $P = Z_5 =$ 5d projective, $V \otimes P = P^{\oplus 3} = 3 \cdot P$




$V^{\otimes 2} \cong 1 \cdot \mathbb{1} \oplus 1 \cdot V \oplus 1 \cdot P$ write $(1, 1, 1)$

$V^{\otimes 3} \leftrightsquigarrow (1, 2, 4)$, $V^{\otimes 4} \leftrightsquigarrow (2, 3, 14)$, $V^{\otimes 5} \leftrightsquigarrow (3, 5, 45)$

$V^{\otimes 6} \leftrightsquigarrow (5, 8, 140)$, $V^{\otimes 7} \leftrightsquigarrow (8, 13, 428)$, $V^{\otimes 8} \leftrightsquigarrow (13, 21, 1297)$

# The nonsem

1.

0.

$n = 20$, ratio 0.99998

**Theorem (Bryant–Kovács ∼1972; correctly interpreted)**

For any finite group $\Gamma$, any field $\mathbb{K}$ and any fin dim $\Gamma$-rep $V$:

$$b_n \sim A \cdot (\dim V)^n \text{ for } A \in \mathbb{R}_{>0}$$

**Example (the picture you just saw)**

$\Gamma = \mathbb{Z}/5\mathbb{Z}$, $\mathbb{K} = \overline{\mathbb{F}_5}$, $V = Z_3 =$ 3d indecomposable, $P = Z_5 =$ 5d projective, $V \otimes P = P^{\oplus 3} = 3 \cdot P$

$$Z_1 \leftrightsquigarrow \begin{pmatrix} 1 \end{pmatrix} \quad Z_2 \leftrightsquigarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad Z_3 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$Z_4 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad Z_5 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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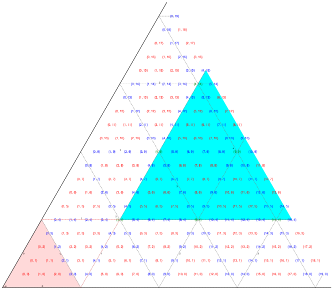
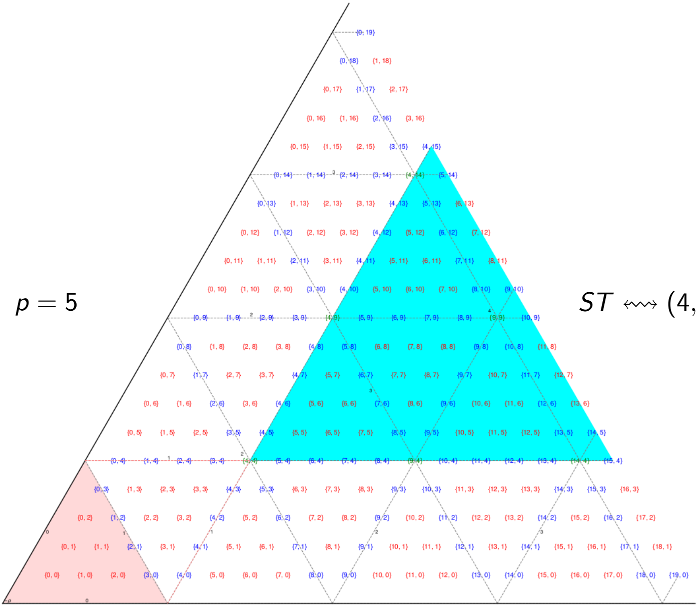
## The nonsemisimple case and Jupiter



- $\Gamma = GL_N$, $\mathbb{K}$ = any field, $V$ = vector rep
- **Theorem (Folklore ∼1970, Andersen ∼2017, Coulembier–Ostrik ∼2023)**
  Essentially all summands of $V^{\otimes n}$ are linked to the Steinberg weight ST
  The Steinberg cone
- $\Gamma$-reps linked to ST have 'known' dimensions
- Non-Steinberg summands 'do not matter' and play the Jupiter argument

# The nonsemisimple case and Jupiter

$p = 5$




$ST \leftrightsquigarrow (4, 4)$

### The embedding trick

**Case 1** $\Gamma$ is an affine group scheme

$$\Rightarrow \Gamma \hookrightarrow GL(V)$$

$$\Rightarrow b_n^{GL(V),V} \leq b_n^{\Gamma,V}$$

$\Rightarrow$ Done

**Case 2** $\Gamma$ is an affine semigroup scheme

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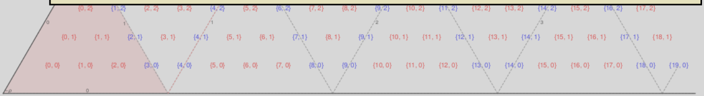
$\Rightarrow$ Done

**Case 3** $\Gamma$ is something super

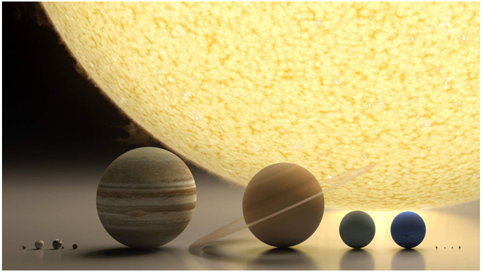
$\Rightarrow$ 'a multiple of $GL_{M|N}$' Now live

$\Rightarrow$ Done

$p =$ … $4)$



## The nonsemisimple case and Jupiter



- $\Gamma = GL_{M|N}$, $P = GL_M \times GL_N$
- **Theorem (Folklore ∼???, Coulembier–Ostrik ∼2023)** $\exists$ constant $A$ such that dim of every indecomposable of $\Gamma$ is bounded by $A \cdot$ dim of an associated indecomposable of $P$
- **Example** $A = 4$ for $GL_{1|1}$, thus every indecomposable $GL_{1|1}$-rep is at most four dimensional since $GL_1 \times GL_1$ is boring
- Hence, the main theorem for $\Gamma$ reduces to $P$ (still the Jupiter argument)

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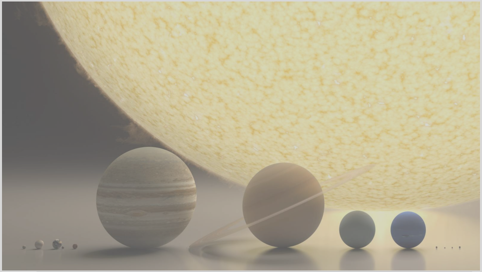

We have now survived the whole proof!

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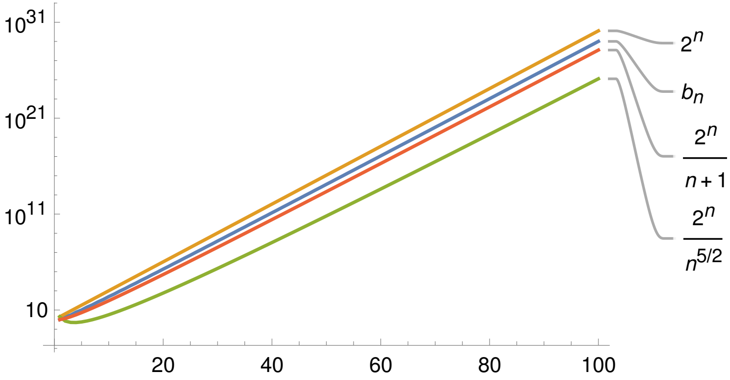
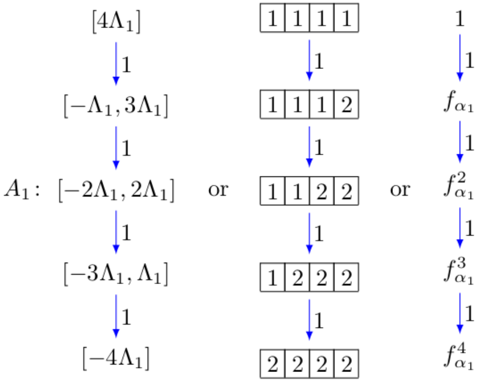



- **Summary** Few summands have high multiplicity, take these and play the Jupiter argument
- As an example: **Theorem (Khovanov–Sitaraman ∼2021)** For $SL_2$ take only summands with highest weight $< \sqrt{n}$ and get $2^n/n^{5/2}$ as a lower bound for $b_n$

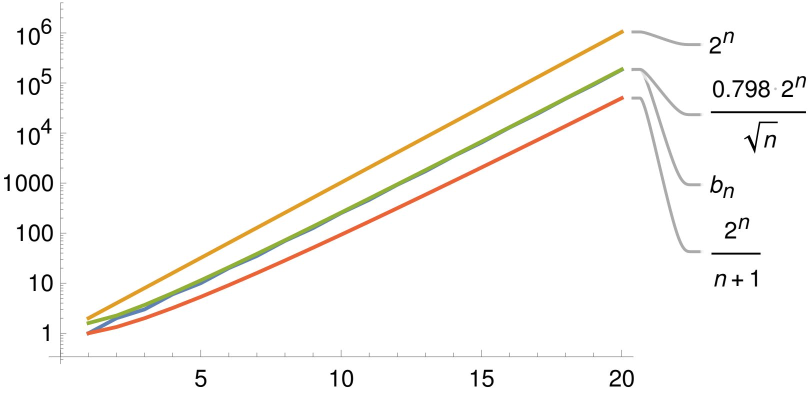
## Results for $SL_2$ beyond Jupiter




$Sym^4\mathbb{C}^2$ as a crystal

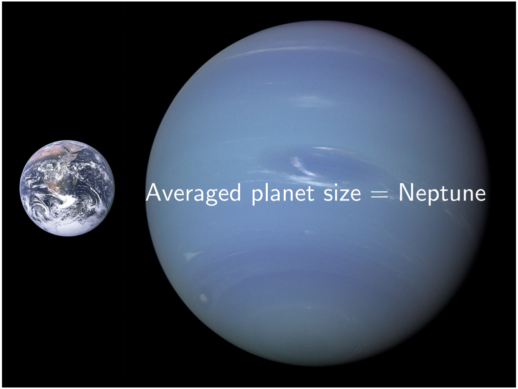
- Simple $SL_2$-reps over $\mathbb{C}$ are 'lines' i.e. $Sym^k\mathbb{C}^2$
- Their character is $q^{-k+1} + q^{-k+3} + ... + q^{k-3} + q^{k-1}$
- In particular, up to parity, they have an unique factor $q^0$ or $q^1$

## Results for $SL_2$ beyond Jupiter



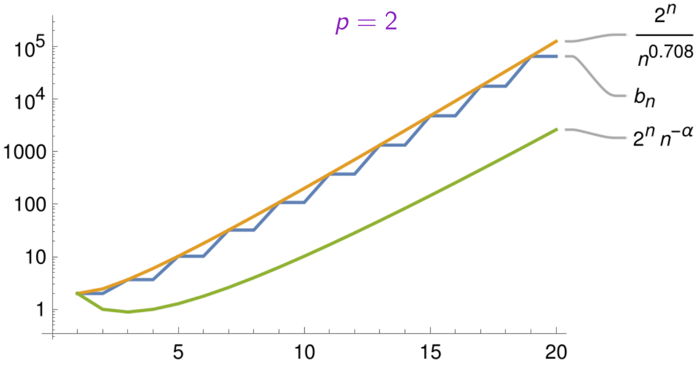

- For $V = \mathbb{C}^2$ the character of $V^{\otimes n}$ is $(q^{-1} + q)^n$
- **Theorem (Folklore ∼1930, Coulembier–Ostrik ∼2023)** $b_n = \binom{n}{\lfloor n/2 \rfloor}$
- Stirling's formula $\Rightarrow$ $b_n \sim \sqrt{2/\pi} \cdot 2^n / \sqrt{n}$ with $\sqrt{2/\pi} \approx 0.798$

## Results for $SL_2$ beyond Jupiter

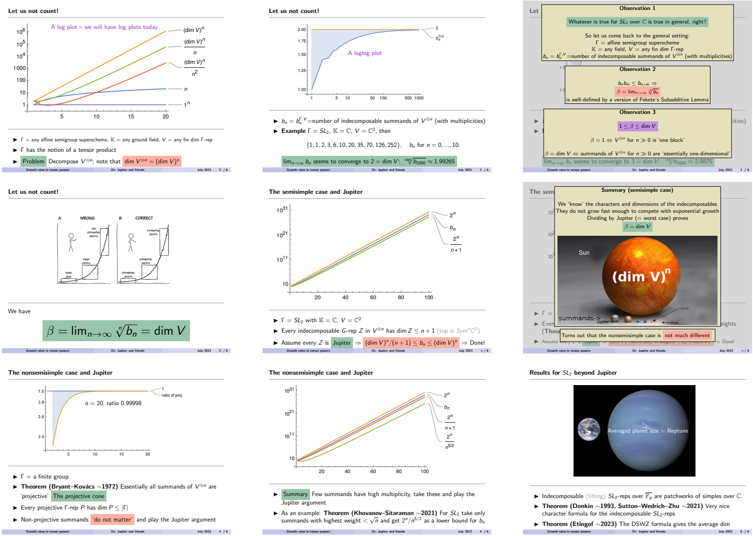



- Indecomposable (tilting) $SL_2$-reps over $\overline{\mathbb{F}_p}$ are patchworks of simples over $\mathbb{C}$
- **Theorem (Donkin ∼1993, Sutton–Wedrich–Zhu ∼2021)** Very nice character formula for the indecomposable $SL_2$-reps
- **Theorem (Etingof ∼2023)** The DSWZ formula gives the average dim

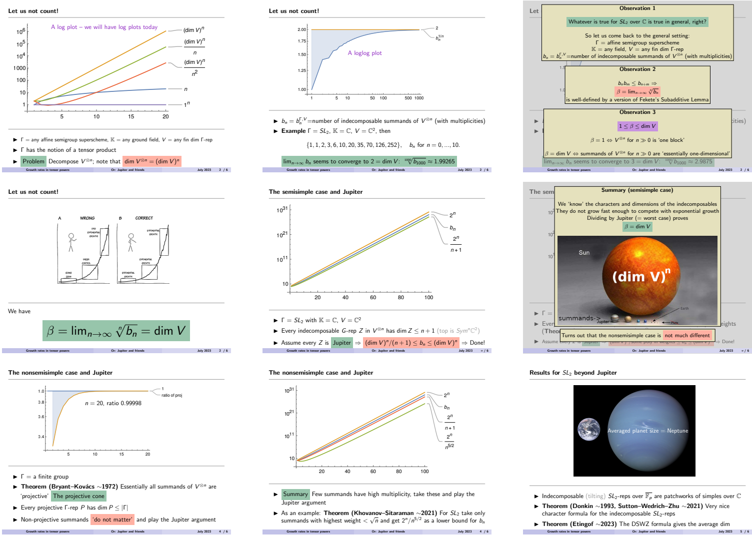
## Results for $SL_2$ beyond Jupiter




- **Theorem (Coulembier–Ostrik ∼2023)** Use the Jupiter value of DSWZ to get a lower bound $2^n n^{-\alpha}$ for $\alpha = 1 + \log_2(p)^{-1}$
- **Conjecture/theorem (Etingof ∼2023)** Use the Neptune value of DSWZ to get the 'real' growth rate, e.g. $\approx 2^n n^{-0.708}$ for $p = 2$




There is still much to do...




Thanks for your attention!