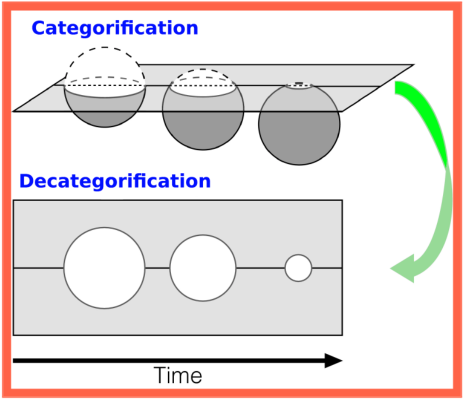
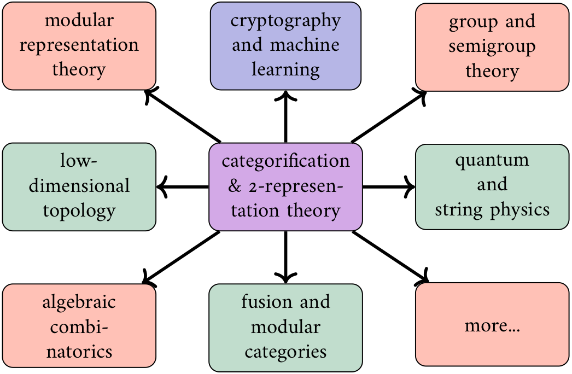
Categorification: objects and their shadows

Or: Homology and actions

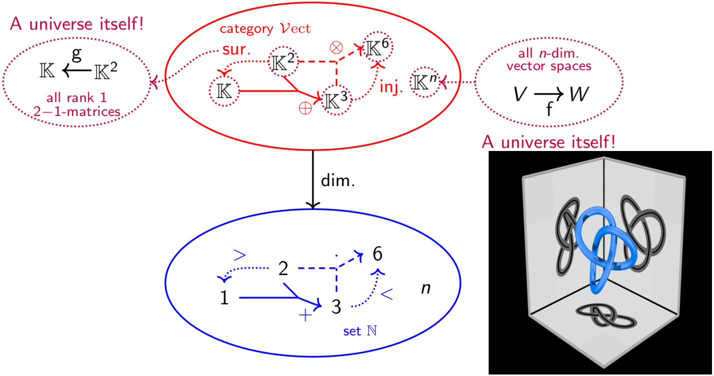


My field of research – categorification in a nutshell



- I am working in categorification and categorical or 2-representation theory
- I am very interested in developing the abstract theory and finding applications
- Applications include the fields above – some are very present at UNSW

My field of research – categorification in a nutshell

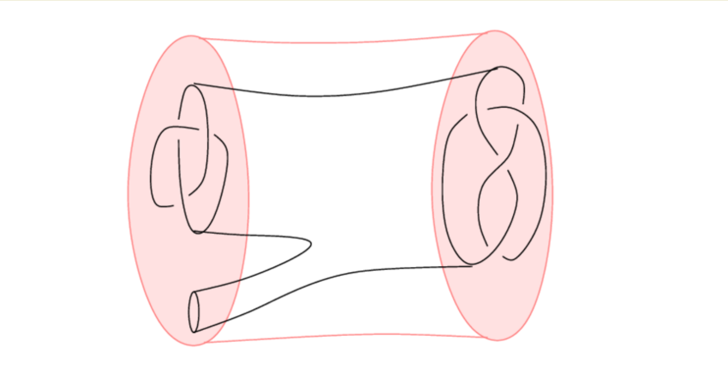


- Categorification = replace set-theoretical structures by category-theoretical ones
- Categorifcation reveals hidden structures “Shadow vs. real object”
- A main upshot Categorification makes connections between fields visible

My fi

Today

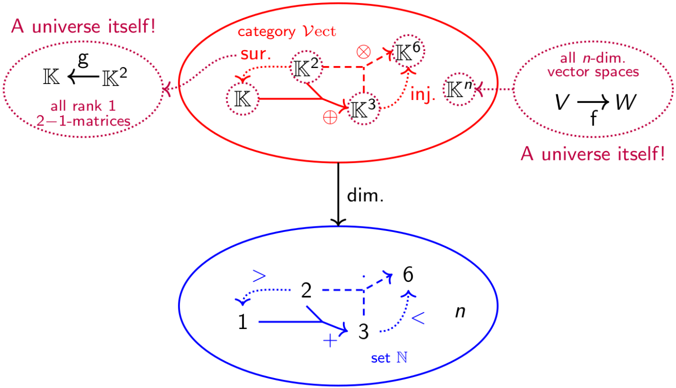
My very biased tour through categorification
with focus on the two aspects “most important”:
Homologies and Actions



Above Knots, links‘ and their categorifications: cobordisms between them

- C ... ones
- C
- A main upshot Categorification makes connections between fields visible

Categorification – homologies

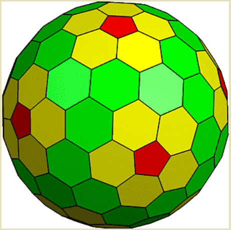


- Categorification, in disguise, is around for Donkey's years
- Example Linear algebra is a categorification of the natural numbers
- And, of course, linear algebra is one of the most successful theories in mathematics

Ca

One of the main first steps in categorification:

Noether, Hopf ∼1925 Singular homology categorifies the Euler characteristic



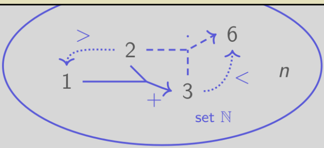
Polyhedron	**E**	**K**	**F**	χ
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Oktahedron	6	12	8	2
Dodekahedron	20	30	12	2
Isokahedron	12	30	20	2

Polyhedron theorem (1736)

Let $P \subset \mathbb{R}^3$ be a convex polyhedron with V vertices, E edges and F faces. Then:

$$\chi = V - E + F = 2.$$

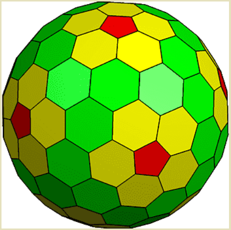
Here χ denotes the Euler characteristic.



- Categorification, in disguise, is around for Donkey's years
- Example Linear algebra is a categorification of the natural numbers
- And, of course, linear algebra is one of the most successful theories in mathematics

One of the main first steps in categorification:

Noether, Hopf ∼1925 Singular homology categorifies the Euler characteristic



Polyhedron	**E**	**K**	**F**	χ
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Oktahedron	6	12	8	2
Dodekahedron	20	30	12	2
Isokahedron	12	30	20	2

Polyhedron theorem (1736)

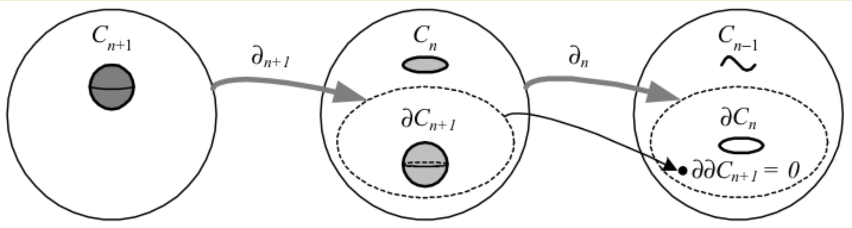
Let $P \subset \mathbb{R}^3$ be a convex polyhedron with V vertices, E edges and F faces. Then:

$$\chi = V - E + F = 2.$$

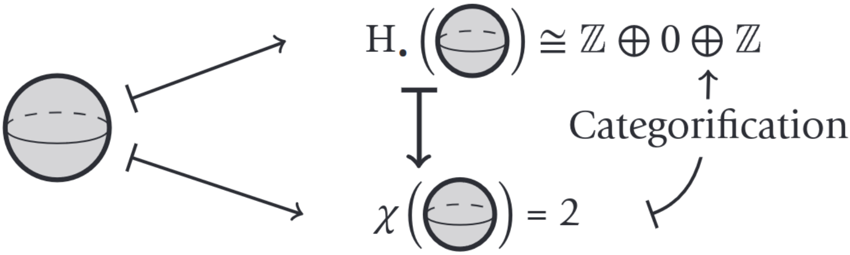
Here χ denotes the Euler characteristic.

Mayer ∼1929 With chain complexes its even nicer:

The Euler characteristic becomes an alternating sum of dimensions



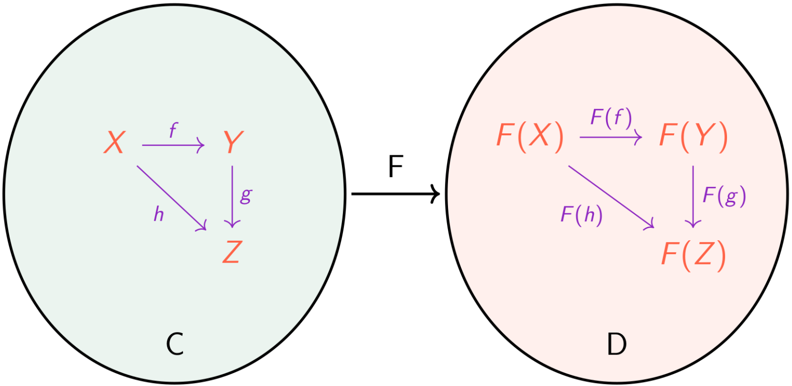
Categorification – homologies



$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim H_i(X)$$

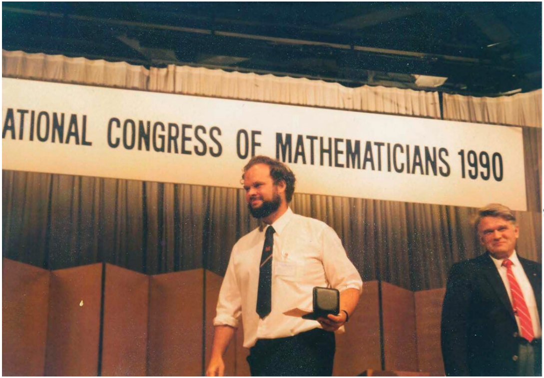
- Example Homological algebra is a categorification of the integers
- Example Simplicial homology categorifies the Euler characteristic
- And, of course, homological algebra is one of the most successful theories in mathematics

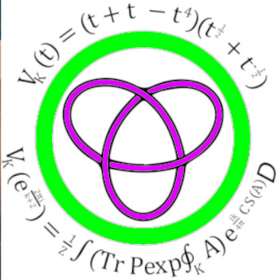
Categorification – homologies



- Homology is a better invariant than the Euler characteristic, and that is good...
- ...but the main upshot is that homology is a functor
- Homology knows about maps as well!

Categorification – homologies

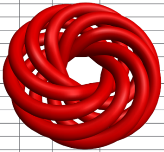




- **Jones** $\sim$**1983** revolutionized knot theory and its ramifications
- Above; Kyoto 1990 Jones walks away with the fields medal
- The Jones polynomial and friends are nowadays among the cornerstones of mathematics and quantum physics

Categorification – homologies

	$j{=}23$	$j{=}25$	$j{=}27$	$j{=}29$	$j{=}31$	$j{=}33$	$j{=}35$	$j{=}37$	$j{=}39$	$j{=}41$	$j{=}43$
$r{=}0$										$\mathbb{Z}$	$\mathbb{Z}$
$r{=}1$											
$r{=}2$										$\mathbb{Z}$	
$r{=}3$										$\mathbb{Z}_2$	$\mathbb{Z}$
$r{=}4$									$\mathbb{Z}$	$\mathbb{Z}$	
$r{=}5$										$\mathbb{Z}$	$\mathbb{Z}$
$r{=}6$								$\mathbb{Z}$	$\mathbb{Z}$		
$r{=}7$								$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}$	
$r{=}8$							$\mathbb{Z}$	$\mathbb{Z}^2$			
$r{=}9$								$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}^2$		
$r{=}10$						$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
$r{=}11$						$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2^2$	$\mathbb{Z}^3$			
$r{=}12$					$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}_5$	$\mathbb{Z}$		
$r{=}13$						$\mathbb{Z}\oplus\mathbb{Z}_2^2$	$\mathbb{Z}^4\oplus\mathbb{Z}_2$	$\mathbb{Z}$			
$r{=}14$					$\mathbb{Z}^2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}$			
$r{=}15$					$\mathbb{Z}_2^2\oplus\mathbb{Z}_7$	$\mathbb{Z}^4\oplus\mathbb{Z}_2$	$\mathbb{Z}^2$				
$r{=}16$				$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}^2\oplus\mathbb{Z}_2$	$\mathbb{Z}$			
$r{=}17$				$\mathbb{Z}_2$	$\mathbb{Z}^3\oplus\mathbb{Z}_2$	$\mathbb{Z}^3$					
$r{=}18$			$\mathbb{Z}$	$\mathbb{Z}^2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}\oplus\mathbb{Z}_4$	$\mathbb{Z}$				
$r{=}19$			$\mathbb{Z}_2$	$\mathbb{Z}^2\oplus\mathbb{Z}_2^2$	$\mathbb{Z}^3$						
$r{=}20$		$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}\oplus\mathbb{Z}_2^2$	$\mathbb{Z}$					
$r{=}21$			$\mathbb{Z}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_7$	$\mathbb{Z}^3\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$						
$r{=}22$		$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_5\oplus\mathbb{Z}_7$	$\mathbb{Z}$						
$r{=}23$		$\mathbb{Z}_2\oplus\mathbb{Z}_7$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$						
$r{=}24$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}_5\oplus\mathbb{Z}_7$	$\mathbb{Z}$							
$r{=}25$		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_5$							
$r{=}26$			$\mathbb{Z}_2\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$							

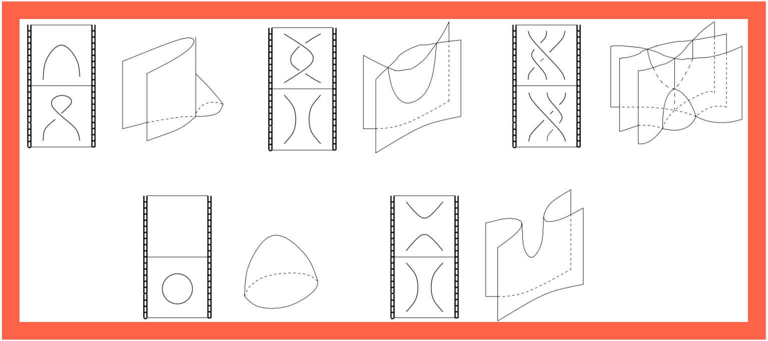


TITLE	CITED BY	YEAR
A categorification of the Jones polynomial M Khovanov	1458	2000

- **Khovanov** $\sim$**1999** There is a categorification of link polynomials using homology
- This breakthrough has citations beyond mathematics
- A goal on many people's research statements: understand Khovanov homology + friends

Categorification – homologies

$$\left\{\begin{matrix}\text{link embeddings in } \mathbb{R}^3 \\ \text{link cobordisms in } \mathbb{R}^3 \times [0,1] \text{ modulo isotopy}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{bigraded vector spaces} \\ \text{homogeneous linear maps}\end{matrix}\right\}$$

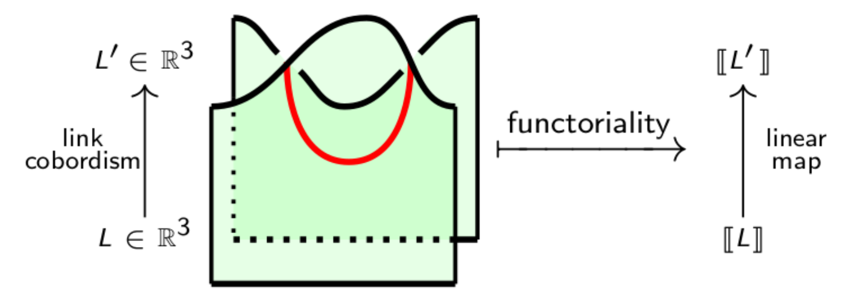


- Homology is a better invariant than the Euler characteristic, and that is good ...
- ...but the main upshot is that homology is a functor **(Joint with Ehrig–Wedrich ∼2018)**
- Link homology knows about cobordisms as well!

Categorification – homologies

{ link embeddings in $\mathbb{R}^3$ } { bigraded vector spaces }

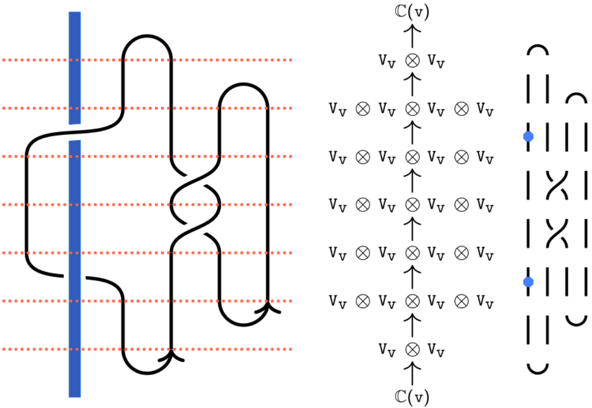
Application | Link homology knows 4d topology



Functoriality is crucial for applications in 4d topology such as the 4d Poincaré conjecture

- Homology is a better invariant than the Euler characteristic, and that is good...
- ...but the main upshot is that homology is a functor **(Joint with Ehrig–Wedrich ∼2018)**
- Link homology knows about cobordisms as well!

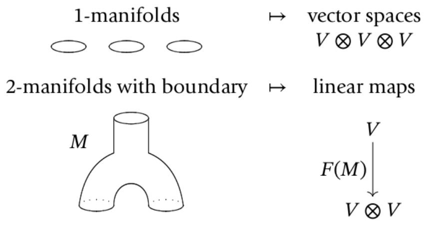
Categorification – actions

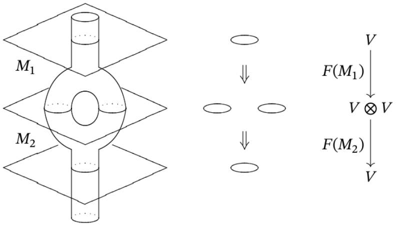


- **Reshetikhin–Turaev ∼1990** gave a rep theoretical construction of Jones-type polynomials
- Roughly, slices of knots are reps and knots equivariant maps
- Upshot Low-dim top "=" rep theory

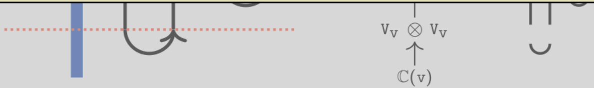
Ca

Around the same time **Crane–Frenkel ∼1994**
asked about the rep theory underlying higher TQFTs





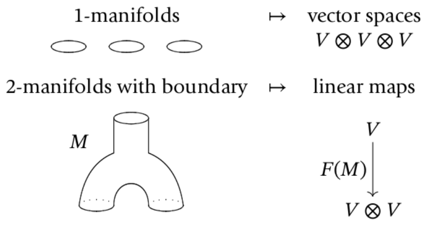
2-dim	3-dim	4-dim
Algebra + extra structure	Category with multiplication + extra structure	2-Category with multiplication + extra structure

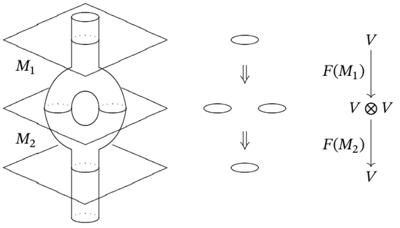


- **Reshetikhin–Turaev ∼1990** gave a rep theoretical construction of Jones-type polynomials
- Roughly, slices of knots are reps and knots equivariant maps
- Upshot Low-dim top “=” rep theory

Ca

Around the same time **Crane–Frenkel** ~**1994**
asked about the rep theory underlying higher TQFTs





2-dim	3-dim	4-dim
Algebra + extra structure	Category with multiplication + extra structure	2-Category with multiplication + extra structure

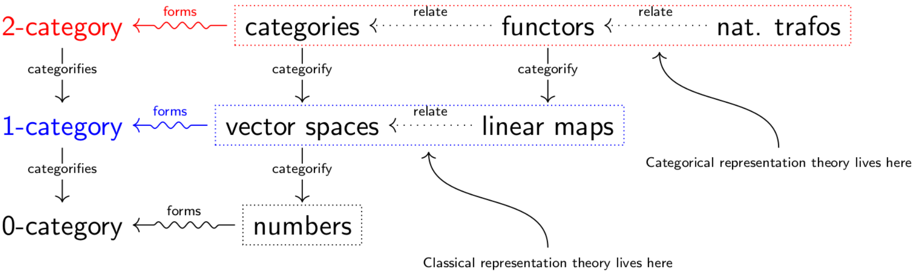
Task ~**90s**

Categorify rep theory itself

Jones polynomial ⟷ Rep theory of quantum $\mathfrak{sl}_2$
↓ ↓
Khovanov homology ⟷ Categorified rep theory of $\mathfrak{sl}_2$

- **Reshetikhin–** ... s-type polynomials
- Roughly, sli
- Upshot Lo

Categorification – actions



- **Chuang–Rouquier∼2004, +others** Cat. Lie group/algebra actions
- **Etingof–Nikshych–Ostrik∼2000, +others** Cat. group actions
- **Joint with Mackaay–Mazorchuk–Miemietz–Zhang ∼2010, +others** Cat. fd algebra actions

Why so many theories? The same is true in the classical case! They all run in parallel, but differ in details.

Categorifica

2-category ← rafos

categorifies

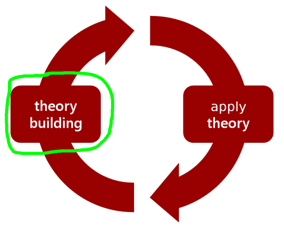
1-category ←

ion theory lives here

categorifies

0-category ← numbers

Classical representation theory lives here



This is theory building so it is hard to nail down one theorem but:

- **Chuang–Rouquier∼2004, +others** Cat. Lie group/algebra actions
- **Etingof–Nikshych–Ostrik∼2000, +others** Cat. group actions
- **Joint with Mackaay–Mazorchuk–Miemietz–Zhang ∼2010, +others** Cat. fd algebra actions

Why so many theories? The same is true in the classical case! They all run in parallel, but differ in details.

Categorifica

2-category ←

categorifies

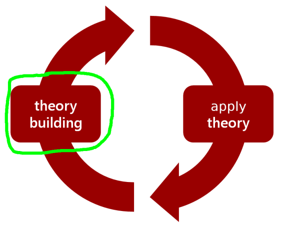
1-category ←

categorifies

0-category ←

rafos

tion theory lives here

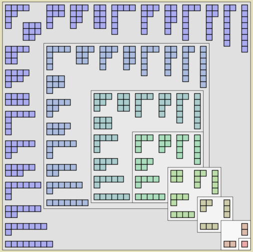


This is theory building so it is hard to nail down one theorem but:

Joint with Mackaay–Mazorchuk–Miemietz–Zhang ∼2021

We worked out the

categorical analog of rep theory of (group rings of) symmetric groups



▶ Ch

▶ Eti

▶ Joi

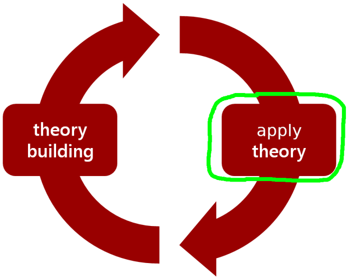
a actions

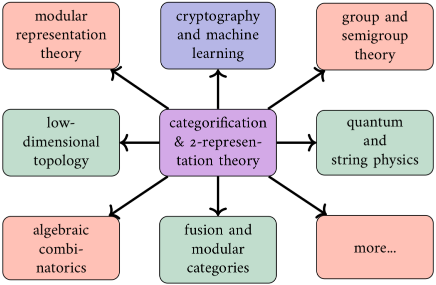
Why so

n in

parallel,

Categorification – actions





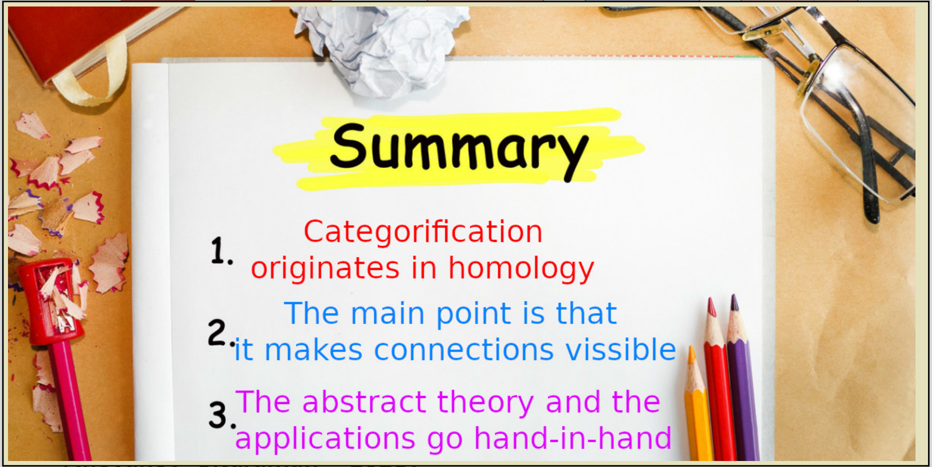
- Application of the theory in all the fields above **Many people ∼2000++**
- Application of the theory in low-dim top **(Joint with Ehrig–Wedrich ∼2018)**
- Application of the theory in monoid-based cryptography **(Joint with Khovanov–Sitaraman ∼2021)**
- Application of the theory in machine learning **(Joint with Gibson–Williamson ∼2023)**

Categorification – actions

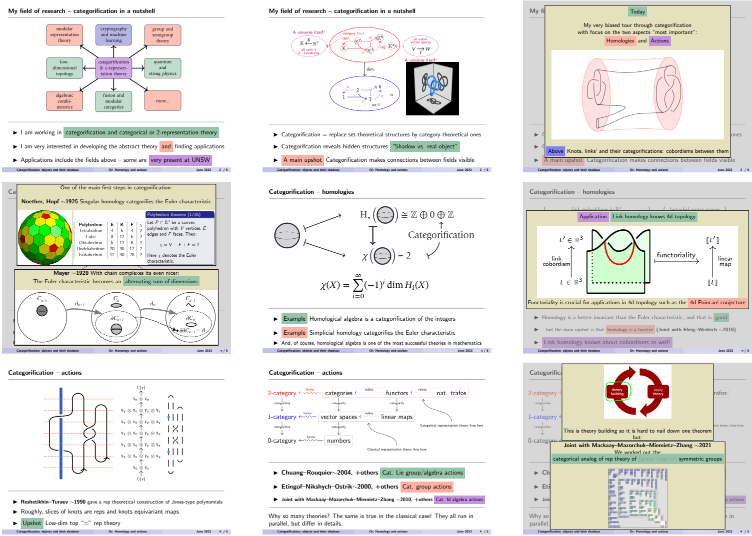
modular representation theory

cryptography and machine learning

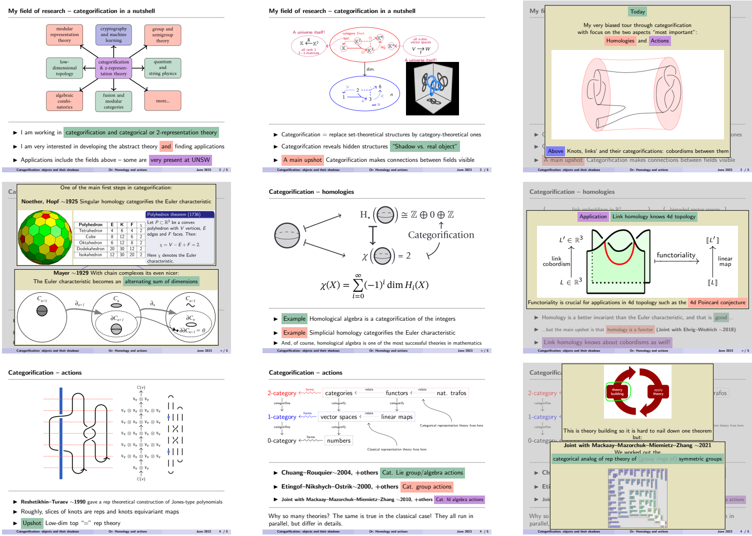
group and semigroup theory



- Application of the theory in machine learning **(Joint with Gibson–Williamson ∼2023)**



There is still much to do...



Thanks for your attention!