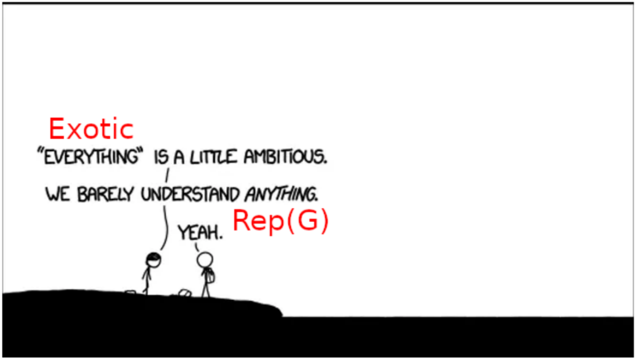
Semisimplifications of tilting modules

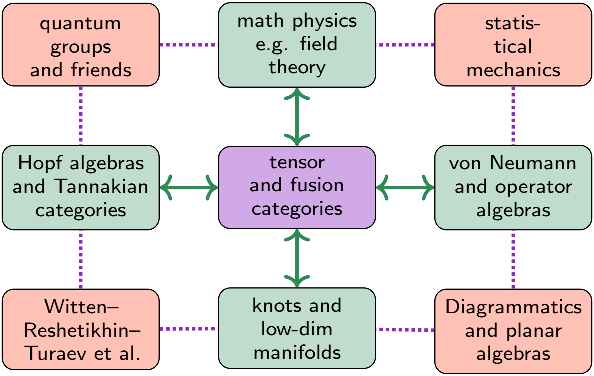
Or: $\mathcal{R}\mathrm{ep}(G)$ everywhere!?

~~Accept~~Change what you cannot ~~change~~accept



I report on work of Elijah Bodish, Jon Brundan, Inna Entova-Aizenbud, Pavel Etingof and Victor Ostrik

Fusion categories



quantum groups and friends

math physics e.g. field theory

statis-tical mechanics

Hopf algebras and Tannakian categories

tensor and fusion categories

von Neumann and operator algebras

Witten–Reshetikhin–Turaev et al.

knots and low-dim manifolds

Diagrammatics and planar algebras

- Tensor categories have been around for Donkey's years
- It took a while until they got formalized; e.g. **Bénabou+MacLane ∼1963**
- The intersection of the above fields gave then birth to the theory of tensor categories; **many people ∼1980++**

Fusion categories

Tensor category $=$ locally finite $\mathbb{K}$-linear abelian rigid monoidal category with bilinear $\otimes$ and End of monoidal unit $\cong \mathbb{K}$

Fusion category $=$ semisimple tensor category

qu
g
and

Hopf algebras and Tannakian categories

tensor and fusion categories

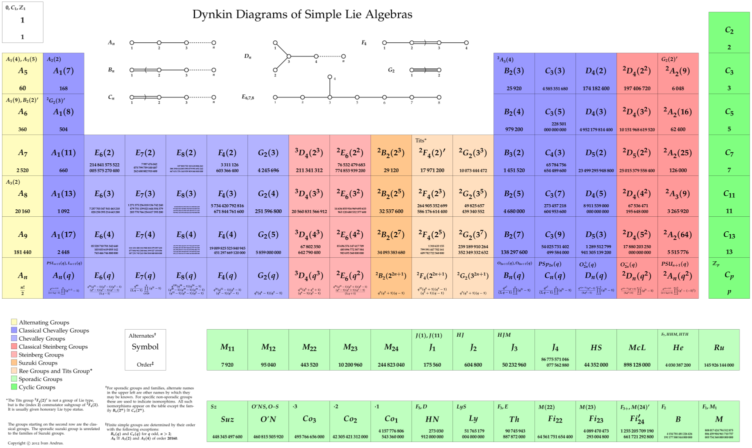
von Neumann and operator algebras

If you do not know what this means in general you are in ~~good~~some company: I do not know either!

We go by examples!

- Tensor ca
- It took a w cLane ∼1963
- The inters eory of tensor categories; **many people** ∼**1980++**

Fusion categories



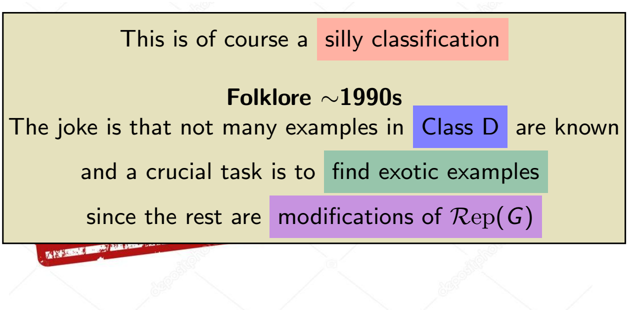
- Classification is a crucial tool in mathematics
- Let me show you a classification of fusion categories

Fusion categories

The classification of fusion categories over $\mathbb{C}$ reads (up to taking products):

- Class A $\mathcal{R}\mathrm{ep}(G)$ for a finite group G or twists
- Class B $\mathcal{V}\mathrm{ec}(G)$ for a finite group G or twists
- Class C Semisimplifications of $\mathcal{R}\mathrm{ep}(U_q(\mathfrak{g}))$ for q-char $\geq$ Coxeter number or twists
- Class D All the rest $=$ exotic examples

Fusion categories



This is of course a silly classification

Folklore ∼1990s

The joke is that not many examples in Class D are known

and a crucial task is to find exotic examples

since the rest are modifications of $\mathcal{R}\mathrm{ep}(G)$

The classification of fusion categories over $\mathbb{C}$ reads (up to taking products):

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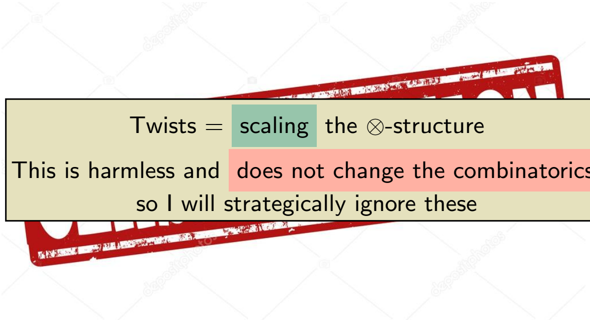
Fusion categories



Twists = scaling the $\otimes$-structure

This is harmless and does not change the combinatorics

so I will strategically ignore these

The classification of fusion categories over $\mathbb{C}$ reads (up to taking products):

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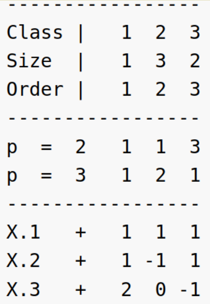
- **Class D** All the rest = exotic examples

Fusion categories

Folklore, Frobenius+Burnside ∼1895++ Class A

Example 1 $\mathcal{R}\mathrm{ep}(1) \cong \mathcal{V}\mathrm{ec}$

Example 2 $\mathcal{R}\mathrm{ep}(S_3)$ is complete determined by the character table of S_3



```
-----------------

Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

Simple objects ⇜⇝ simple characters

$\otimes$ ⇜⇝ multiplication of simple characters

e.g. $\chi_3^{\otimes 2} = (4,0,1) = \chi_1 + \chi_2 + \chi_3$

hom-spaces are determined by Schur's lemma: $\hom_{\mathcal{R}\mathrm{ep}(S_3)}(\chi_i,\chi_j) \cong \delta_{i,j}\mathbb{C}$

The cl

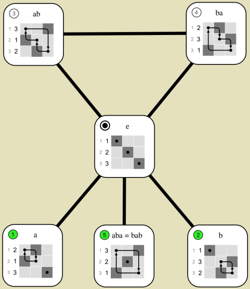
- C
- C
- C twists
- Class D All the rest = exotic examples

Fusion categories

Folklore, Galois+Cauchy+Cayley ∼1832++ Class B

Example 1 $\mathcal{V}\mathrm{ec}(1) \cong \mathcal{V}\mathrm{ec}$

Example 2 $\mathcal{V}\mathrm{ec}(S_3)$ is complete determined by S_3 itself



Simple objects ⇜⇝ elements of S_3

$\otimes$ ⇜⇝ multiplication of elements

e.g. $(123)^{\otimes 2} = (132)$

hom-spaces are determined by Schur's lemma: $\mathrm{hom}_{\mathcal{V}\mathrm{ec}(S_3)}(g, h) \cong \delta_{g,h}\mathbb{C}$

The cla :

- C
- C
- C r twists
- Class D All the rest = exotic examples

Fusion cate

Folklore, Verlinde+Turaev+Andersen ~1988++ Class C

A bit more difficult then the first two classes
I will explain them momentarily
These depend on $q \in \mathbb{C}, q^{\ell} = 1$ for ℓ not too small

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A bit more difficult then the first two classes
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Class D

I essentially only know two types of examples

Class D.a Exotic subfactors
à la e.g. **Haagerup $\sim$1993**

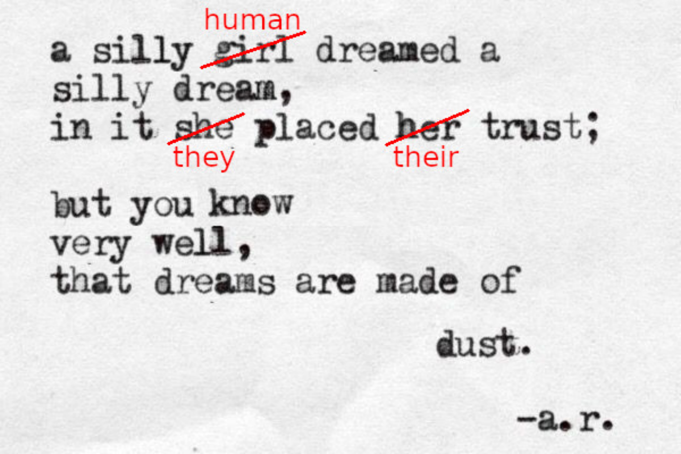
Class D.b Exotic asymptotic Hecke categories
à la e.g. **Lusztig+Du Cloux+Alvis+Elias–Williamson $\sim$1987++**

We are looking for more examples!

The classi ts):

- Clas
- Clas
- Clas r or twists
- Class D All the rest = exotic examples

Fusion categories



a silly ~~girl~~ human dreamed a
silly dream,
in it ~~she~~ they placed ~~her~~ their trust;

but you knew
very well,
that dreams are made of
dust.

-a.r.

My hope: for ℓ small Class C could give some exotic fusion categories

Spoiler: I was completely wrong!

The cla

- C
- C
- Class C Semisimplifications of $\mathcal{R}\mathrm{ep}(U_q(\mathfrak{g}))$ for q-char $\geq$ Coxeter number or twists
- Class D All the rest = exotic examples

The Verlinde categories

$$Z_1 \rightsquigarrow \begin{pmatrix} \boxed{1} \end{pmatrix} \quad Z_2 \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 \\ \boxed{1} & \boxed{1} \end{pmatrix} \quad Z_3 \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & 0 \\ \boxed{1} & \boxed{1} & 0 \\ 0 & \boxed{1} & \boxed{1} \end{pmatrix}$$

$$Z_4 \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ \boxed{1} & \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} & \boxed{1} \end{pmatrix} \quad Z_5 \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ \boxed{1} & \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} & \boxed{1} \end{pmatrix}$$

- $G = \mathbb{Z}/5\mathbb{Z}$ has five indecomposables Z_1 to Z_5 over $\mathbb{K} = \overline{\mathbb{F}_p}$
- They are given by sending 1 to an indecomposable Jordan block
- They have dimensions 1 to 5

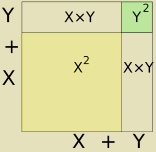
The Verlinde ca

$Z_1 \leftrightsquigarrow$

$Z_4 \leftrightsquigarrow$

Comment 1

Where are the other simples of $\mathbb{Z}/5\mathbb{Z}$ over $\mathbb{C}$? Well:



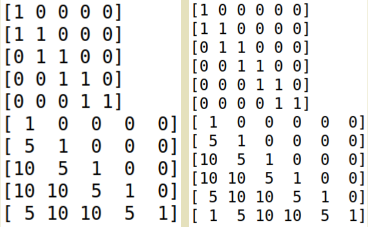
Fresh's dream: $(X+Y)^5 \equiv X^5 + Y^5 \bmod p$
so 1 is the only 5th root of unity in characteristic 5

Comment 2

Why maximally a 5x5 block? Well:

M top

M^5 bottom



```
[1 0 0 0 0]
[1 1 0 0 0]
[0 1 1 0 0]
[0 0 1 1 0]
[0 0 0 1 1]
[ 1  0  0  0  0]
[ 5  1  0  0  0]
[10  5  1  0  0]
[10 10  5  1  0]
[ 5 10 10  5  1]
```

```
[1 0 0 0 0 0]
[1 1 0 0 0 0]
[0 1 1 0 0 0]
[0 0 1 1 0 0]
[0 0 0 1 1 0]
[0 0 0 0 1 1]
[ 1  0  0  0  0  0]
[ 5  1  0  0  0  0]
[10  5  1  0  0  0]
[10 10  5  1  0  0]
[ 5 10 10  5  1  0]
[ 1  5 10 10  5  1]
```

So bigger Jordan blocks will not satisfy $M^5 = 1$

- $G = \mathbb{Z}/5\mathbb{Z}$ h
- They are give
- They have di

The Verlinde categories

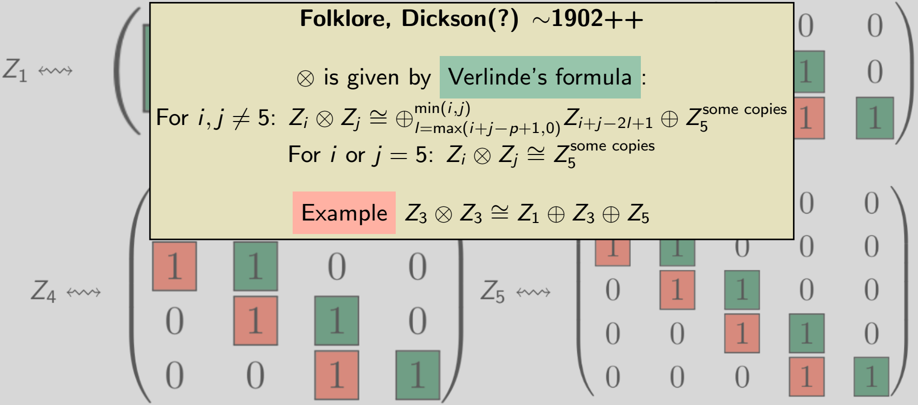
Folklore, Dickson(?) ~1902++

$\otimes$ is given by Verlinde's formula:

For $i, j \neq 5$: $Z_i \otimes Z_j \cong \oplus_{l=\max(i+j-p+1,0)}^{\min(i,j)} Z_{i+j-2l+1} \oplus Z_5^{\text{some copies}}$

For i or $j = 5$: $Z_i \otimes Z_j \cong Z_5^{\text{some copies}}$

Example $Z_3 \otimes Z_3 \cong Z_1 \oplus Z_3 \oplus Z_5$



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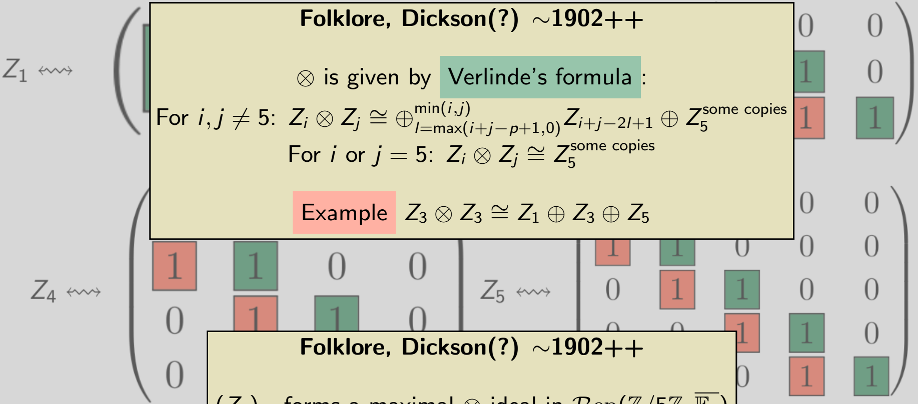
The Verlinde categories



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For i or $j=5$: $Z_i \otimes Z_j \cong Z_5^{\text{some copies}}$

Example $Z_3 \otimes Z_3 \cong Z_1 \oplus Z_3 \oplus Z_5$

Folklore, Dickson(?) ∼1902++

$(Z_5)_\otimes$ forms a maximal $\otimes$-ideal in $\mathcal{R}\mathrm{ep}(\mathbb{Z}/5\mathbb{Z},\overline{\mathbb{F}_5})$

Verlinde category $\mathcal{V}\mathrm{er}_5 := \mathcal{R}\mathrm{ep}(\mathbb{Z}/5\mathbb{Z},\overline{\mathbb{F}_5})/(Z_5)_\otimes$

And of course we also get $\mathcal{V}\mathrm{er}_p$ for any prime p

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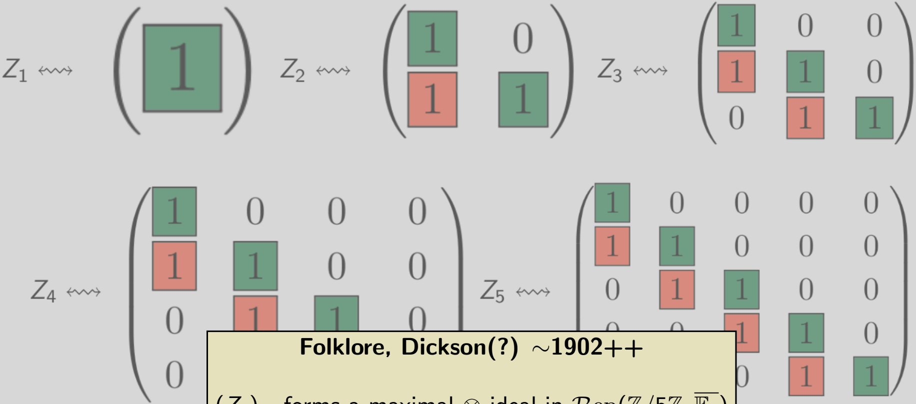

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$$Z_4 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & & & \end{pmatrix} \quad Z_5 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ & & 0 & 1 & 1 \end{pmatrix}$$

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The Verlinde categories

Folklore, Weyl(?) $\sim$1930++

$\mathcal{V}\mathrm{er}_p$ can be alternatively constructed using $SL_2(\mathbb{F}_p)$ or $SL_2(\overline{\mathbb{F}_p})$

More momentarily

$Z_1 \leftrightsquigarrow$

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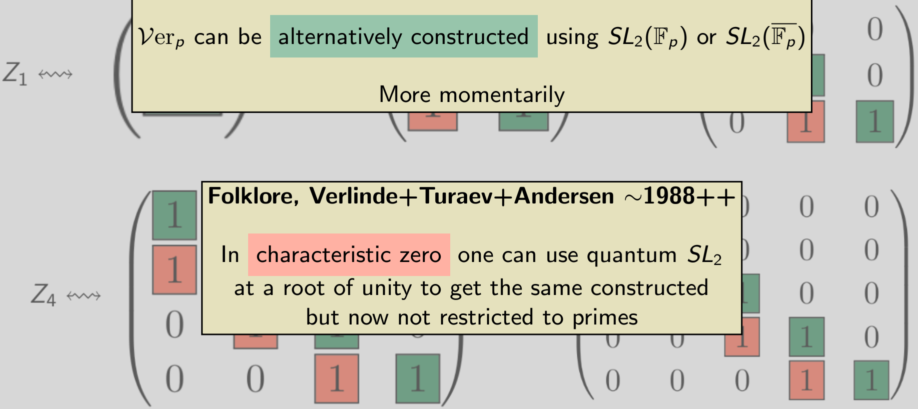

$Z_1 \leftrightsquigarrow$

$Z_4 \leftrightsquigarrow$

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More momentarily

Folklore, Verlinde+Turaev+Andersen $\sim$1988++

In characteristic zero one can use quantum SL_2 at a root of unity to get the same constructed but now not restricted to primes

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More momentarily

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$Z_4 \leftrightsquigarrow$

Folklore, Weyl(?) ∼1930++, Verlinde+Turaev+Andersen ∼1988++

It then makes sense to define
Verlinde categories for SL_n, O_n, SP_n etc.

More momentarily

- C
- Tl
- They have dimensions 1 to 5

The V

$Z_1 \leftrightsquigarrow$ $\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\Bigg)$

Verlinde categories $\leftrightsquigarrow$ calculus of indecomposable Jordan blocks for $\overline{\mathbb{F}_p}$

There is also a realization using SL_2

$$Z_4 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad Z_5 \leftrightsquigarrow \begin{pmatrix} & & & & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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Verlinde categories $\rightsquigarrow$ calculus of indecomposable Jordan blocks for $\overline{\mathbb{F}_p}$

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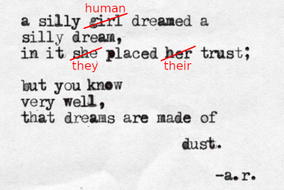
Z_4 0


a silly ~~girl~~ human dreamed a
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but you knew
very well,
that dreams are made of
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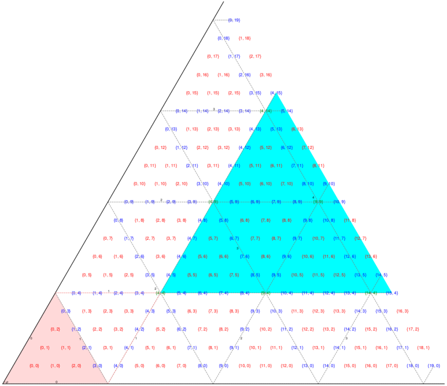
-a.r.

So far we had only $\mathcal{R}\mathrm{ep}(G)$-type categories:
$\mathcal{R}\mathrm{ep}(G)$, $\mathcal{V}\mathrm{ec}(G)$ = dual of $\mathcal{R}\mathrm{ep}(G)$
$\mathcal{V}\mathrm{er}_p$ = modification of $\mathcal{R}\mathrm{ep}(G) \Leftarrow$ semisimplifications of $\mathcal{R}\mathrm{ep}(U_q(\mathfrak{g}))$ (up next)

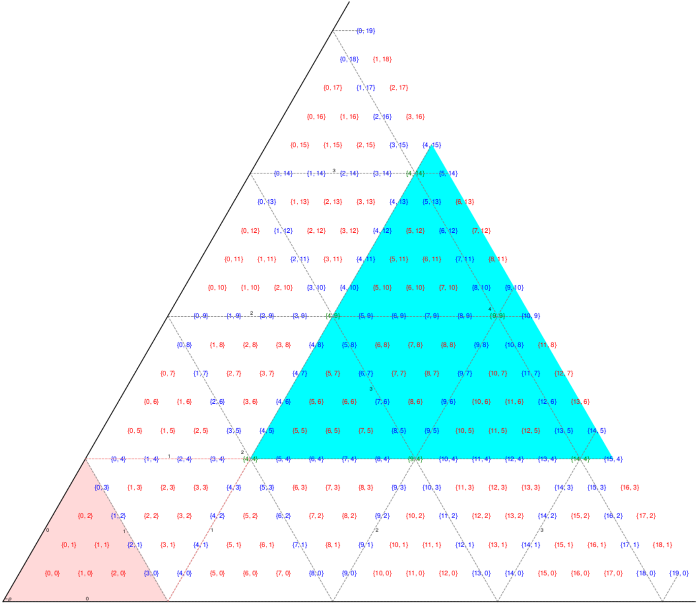
We are still searching for something new!

The higher rank Verlinde categories



- $SL_3(\overline{\mathbb{F}_p})$ has the same classification of simples as $SL_3(\mathbb{C})$ (SL_n is similar)
- The simples correspond to $(a_1, a_2) \in \mathbb{N}^2$
- We put them in a p-scaled alcove picture for $p \geq 3$

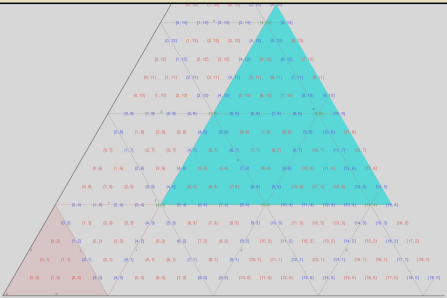
The higher rank Verlinde categories



The higher rank

Higher rank Verlinde rules **Verlinde ∼1988**

For X, Y in the pink triangle (fundamental alcove):
$X \otimes Y =$ things in the pink triangle $\oplus$ other stuff
Throw away the other stuff



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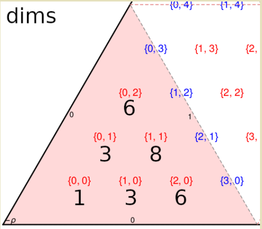
$X \otimes Y =$ things in the pink triangle $\oplus$ other stuff
Throw away the other stuff

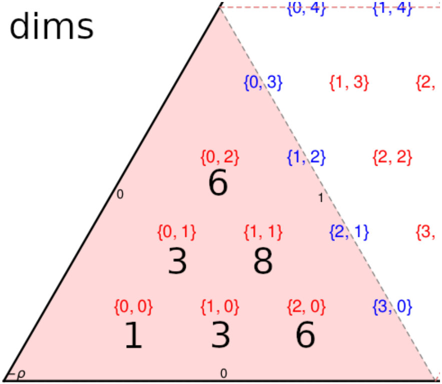
Higher rank Verlinde categories **Turaev–Andersen ∼1992**

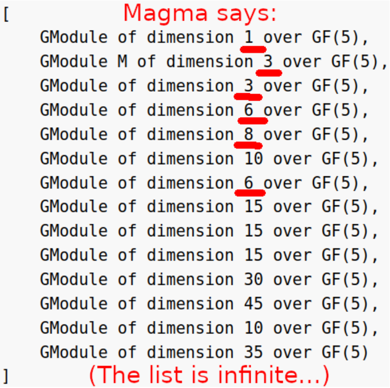
Corresponding fusion categories (also for O, SP etc.) can be constructed
by semisimplifying tilting module categories
Their simples are indexed by analogs of the pink triangle



If you do not know what that means: no worries!
We will see an alternative construction momentarily

The higher rank Verlinde categories





Magma says:

```
[
    GModule of dimension 1 over GF(5),
    GModule M of dimension 3 over GF(5),
    GModule of dimension 3 over GF(5),
    GModule of dimension 6 over GF(5),
    GModule of dimension 8 over GF(5),
    GModule of dimension 10 over GF(5),
    GModule of dimension 6 over GF(5),
    GModule of dimension 15 over GF(5),
    GModule of dimension 15 over GF(5),
    GModule of dimension 15 over GF(5),
    GModule of dimension 30 over GF(5),
    GModule of dimension 45 over GF(5),
    GModule of dimension 10 over GF(5),
    GModule of dimension 35 over GF(5)
]
```

(The list is infinite...)

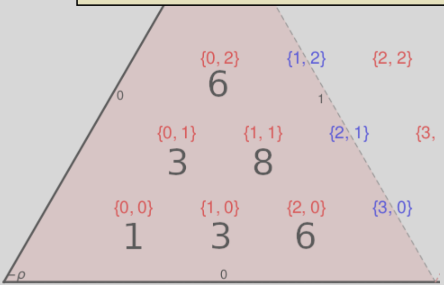
- Take the finite group $SL_3(\mathbb{F}_p)$ and reps over $\mathbb{F}_p$ for $p \geq 3$ (SL_n is similar)
- Say $p = 5$, then the category $\otimes$-generated by $V = \mathbb{F}_5^3$ has only six reps of dimension coprime to 5; these have dims $1, 3, 3, 6, 6, 8$

The higl

Higher rank Verlinde categories - $\mathbb{F}_p$ **Folklore, Weyl(?) ∼1930++**

For $G = SL_n, O_n, SP_{2n}, \dots$ and a fixed prime $p \geq$Coxeter number
the higher rank Verlinde categories $\mathcal{V}\mathrm{er}_p(G)$ over $\mathbb{F}_p$
can be constructed by semisimplifying $\mathcal{R}\mathrm{ep}(G(\mathbb{F}_p), \mathbb{F}_p)$
semisimplifying $=$ kill morphisms of trace zero

dim



```
                                    5),
                                    F(5),
                                    5),
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(The list is infinite...)

- Take the finite group $SL_3(\mathbb{F}_p)$ and reps over $\mathbb{F}_p$ for $p > 3$ (SL_n is similar)

Higher rank Verlinde categories - general fields **Verlinde+Turaev+Andersen ∼1988++**

Over arbitrary fields one can do an analog construction
using tilting modules of quantum group

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 over GF(5),
 over GF(5),
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```

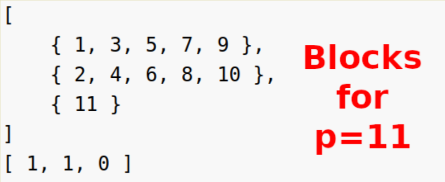
nite...)

Higher rank Verlinde categories - $\mathbb{F}_p$ **Folklore, Weyl(?) ~1930++**

For $G = SL_n, O_n, SP_{2n}, \ldots$ and a fixed prime $p \geq$Coxeter number
the higher rank Verlinde categories $\mathcal{V}\mathrm{er}_p(G)$ over $\mathbb{F}_p$
can be constructed by semisimplifying $\mathcal{R}\mathrm{ep}(G(\mathbb{F}_p), \mathbb{F}_p)$
semisimplifying = kill morphisms of trace zero

Example (back to SL_2)

The p-Sylow subgroups of $SL_2(\mathbb{F}_p)$ are $\mathbb{Z}/p\mathbb{Z}$
$SL_2(\mathbb{F}_p)$ has two blocks with defect group $\mathbb{Z}/p\mathbb{Z}$
$\mathcal{V}\mathrm{er}_p(SL_2)$ = semisimplification of these two block



```
[
    { 1, 3, 5, 7, 9 },
    { 2, 4, 6, 8, 10 },
    { 11 }
]
[ 1, 1, 0 ]
```

Blocks for p=11

- Take the fi

(SL_n is similar)

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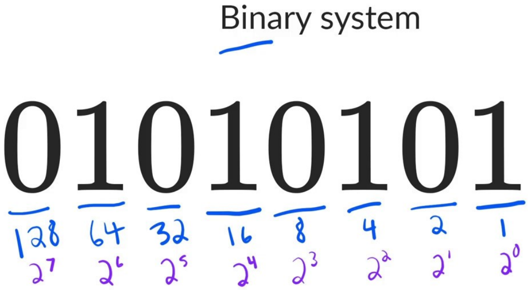
The higher rank Verlinde categories



- **Brundan–Entova-Aizenbud–Etingof–Ostrik ∼2020** For general p we have

$$\mathcal{V}\mathrm{er}_p(GL_n) \cong \mathcal{V}\mathrm{er}_p(GL_{n_k}) \boxtimes \ldots \boxtimes \mathcal{V}\mathrm{er}_p(GL_{n_0}) \text{ (as sym. fusion cats)}$$

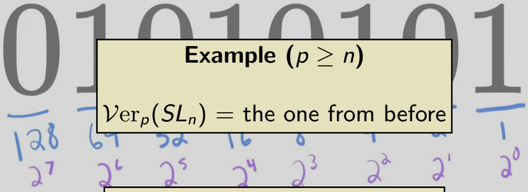
$$n = n_k p^k + \ldots + n_0 p^0, n_i \in \{0, \ldots, p-1\}$$

- **Bodish ∼2023** Similarly for O_n and SP_{2n} (not quite done)

The higher rank Verlinde categories

Mild catch

This is all non-quantum and characteristic p but there are also quantum versions over any field



Example ($p \geq n$)

$\mathcal{V}\mathrm{er}_p(SL_n)$ = the one from before

Example ($p = 2$)

$\mathcal{V}\mathrm{er}_2(SL_n)$ = a product of $\mathcal{V}\mathrm{ec}$

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Example ($p = 3$)

$\mathcal{V}\mathrm{er}_3(SL_n)$ = a product of $\mathcal{V}\mathrm{ec}$ $(n_i = 0, 1)$ and $\mathcal{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$ $(n_i = 2)$

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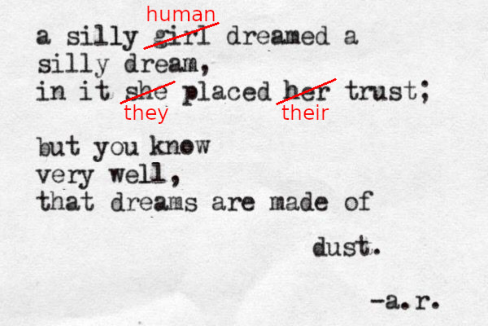
The high

- **Brun**

we have

cats)

- **Bodi**



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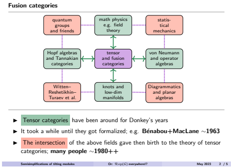
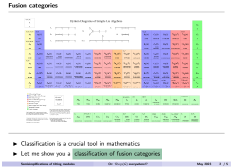


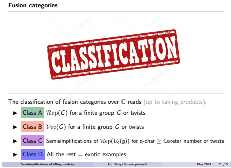
Fusion categories

- Tensor categories have been around for Donkey's years
- It took a while until they got formalized; e.g. **Bénabou+MacLane ∼1963**
- The intersection of the above fields gave then birth to the theory of tensor categories; **many people ∼1980++**



Fusion categories

- Classification is a crucial tool in mathematics
- Let me show you a classification of fusion categories



Fusion categories

The classification of fusion categories over $\mathbb{C}$ reads (up to taking products):

- Class A $\mathcal{R}ep(G)$ for a finite group G or twists
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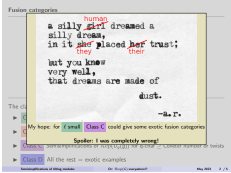

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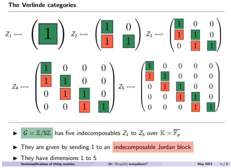

The Verlinde categories

$$Z_1 \rightsquigarrow \begin{pmatrix}1\end{pmatrix} \quad Z_2 \rightsquigarrow \begin{pmatrix}1&0\\1&1\end{pmatrix} \quad Z_3 \rightsquigarrow \begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}$$

$$Z_4 \rightsquigarrow \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix} \quad Z_5 \rightsquigarrow \begin{pmatrix}1&0&0&0&0\\1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\end{pmatrix}$$

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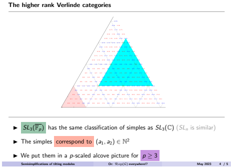

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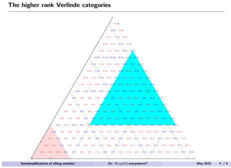

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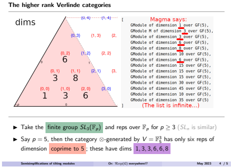

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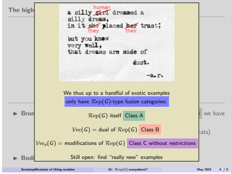

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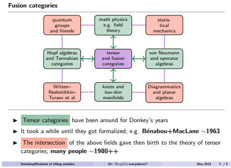

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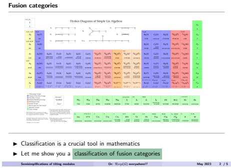
Semisimplifications of tilting modules Or: $\mathcal{R}ep(G)$ everywhere!? May 2023 2 / 5



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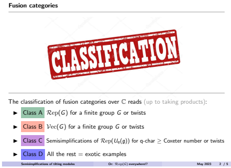

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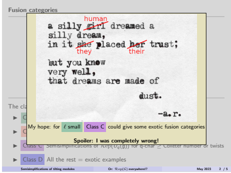

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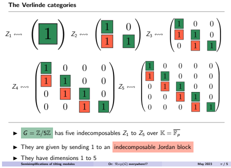

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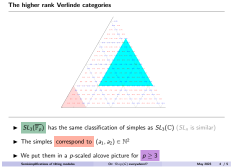

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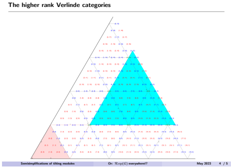

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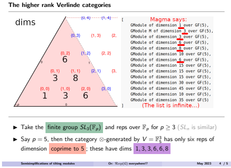

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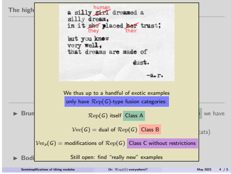

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Thanks for your attention!