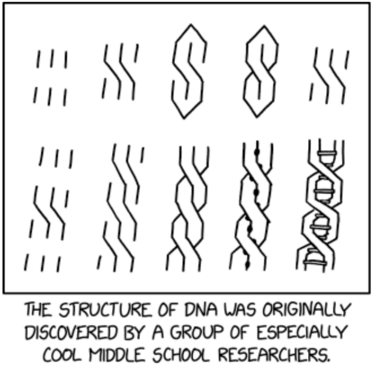
Representations of braids and Howe duality

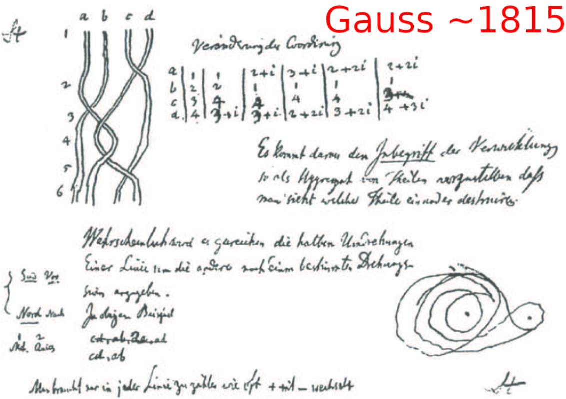
Or: Large = good

~~Accept~~Change what you cannot ~~change~~accept



I report on work of Abel Lacabanne and Pedro Vaz

Braids and representations



- Braid groups have been around for Donkey's years
- It took a while until braid got formalized; e.g. **Artin ∼1925**
- What makes them so tantalizing is that they are in the intersection of topology and algebra, and difficult and easy at the same time

Braids and representations

SageMath

```
B = BraidGroup(3)
b = B([1, 2, 1])
b.LKB_matrix(variables='q,t')
[             0 -q^4*t + q^3*t         -q^4*t]
[             0        -q^3*t              0]
[        -q^2*t  q^3*t - q^2*t              0]
c = B([2, 1, 2])
c.LKB_matrix(variables='q,t')
[             0 -q^4*t + q^3*t         -q^4*t]
[             0        -q^3*t              0]
[        -q^2*t  q^3*t - q^2*t              0]
```

- **Laurence–Krammer–Bigelow (LKB)** $\sim$ **2001** Braid groups are linear
- I.e. there is a way to associate matrices $M(\beta)$ to braids β such that

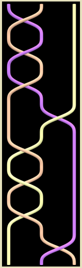
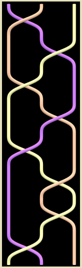
$$\beta = \gamma \Leftrightarrow M(\beta) = M(\gamma)$$

```
B = Bra
b = B([
b.LKB_m
[                 -q^4*t]
[                      0]
[                      0]
c = B([
c.LKB_m
[                 -q^4*t]
[                      0]
[                      0]
```

SageMath

This is pretty darn awesome!

This solves the recognition problem of braids, e.g.

 =  ?

which we can check in SageMath:

Type some Sage code below and press Evaluate.

```
B=BraidGroup(3)
b=B([2,-1,-1,-1,2,1,1,1])
Bb=b.LKB_matrix()
c=B([1,2,2,1,2,1,2,1])
Bc=c.LKB_matrix()
Bb==Bc
```

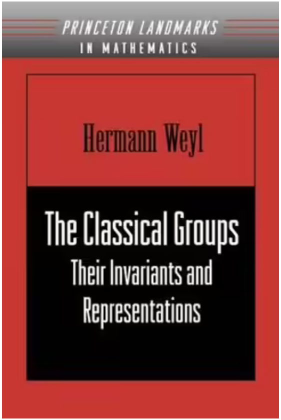
Evaluate

```
False
```

- **Laurence–K** ... ps are linear
- I.e. there is ... uch that

Braids and representations

Weyl $\sim$**1946**:



- **Howe** $\sim$ **1989++** Schur–Weyl duality approach to classical invariant theory
- Howe took Weyl's, quote, "wonderful and terrible book" on classical invariant theory and reformulated it in terms of double centralizers

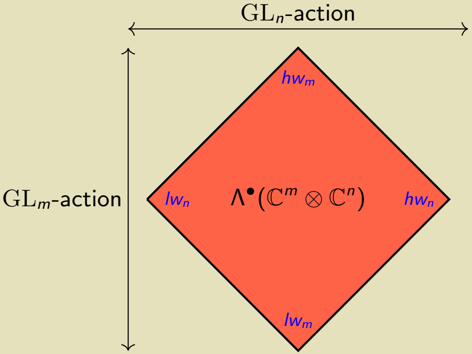
Example **Howe** $\sim$**1989**

There are two commuting actions on the exterior algebra

$$\mathrm{GL}_m \circlearrowright \Lambda^{\bullet}(\mathbb{C}^m \otimes \mathbb{C}^n) \circlearrowleft \mathrm{GL}_n$$

that generate each other's centralizer, i.e.

$\mathbb{C}\mathrm{GL}_m \twoheadrightarrow \mathrm{End}_{\mathrm{GL}_n}\big(\Lambda^{\bullet}(\mathbb{C}^m \otimes \mathbb{C}^n)\big)$ and vice versa with picture:



- **Howe** ... ariant theory
- Howe t... ssical invariant theory and reformulated it in terms of double centralizers

Example **Howe** $\sim$**1989** continued

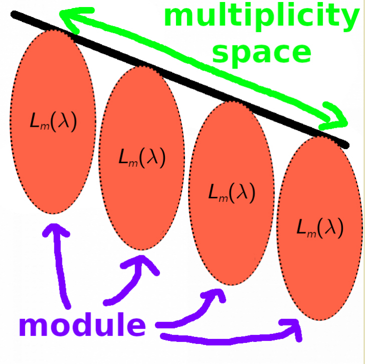
Adamovich–Rybnikov $\sim$**1996** argued that this version of Howe duality is tilting theory
We won't need this
but this essential means that you can tilt your head and things look the same

- Howe took Weyl's, quote, "wonderful and terrible book" on classical invariant theory and reformulated it in terms of double centralizers

Example Howe ∼1989 continued

Howe showed even more: the copies of the simple GL_m-module $L_m(\lambda)$ appearing in $\Lambda^\bullet(\mathbb{C}^m \otimes \mathbb{C}^n)$ form a GL_n-module $L_n(\lambda^T)$ and vice versa with picture:

$\bigoplus_\lambda$

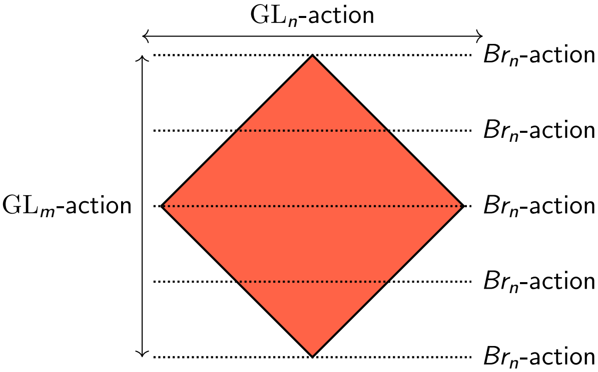


GL_m-GL_n-bimodule decomposition $\Lambda^\bullet(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_\lambda L_n(\lambda) \otimes L_m(\lambda^T)$

ory

invariant theory and reformulated it in terms of double centralizers

Braids and representations

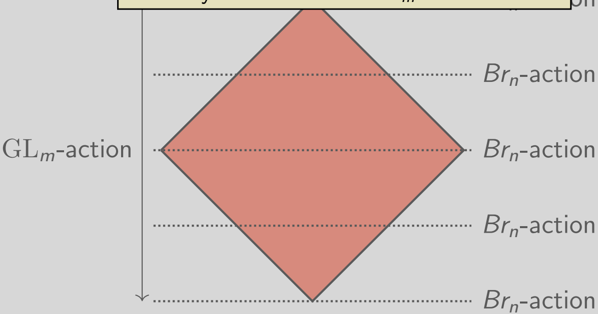


- Let us focus on GL_m-GL_n dualities
- Example **Howe ∼1989** continued Every GL_m-weight space carries an action of the n-strand braid group Br_n Braid reps

Braids and represe

Upshot

We get many Br_n-reps
and they are related to GL_m-combinatorics



- Let us focus on GL_m-GL_n dualities
- Example **Howe ∼1989** continued Every GL_m-weight space carries an action of the n-strand braid group Br_n Braid reps

Upshot

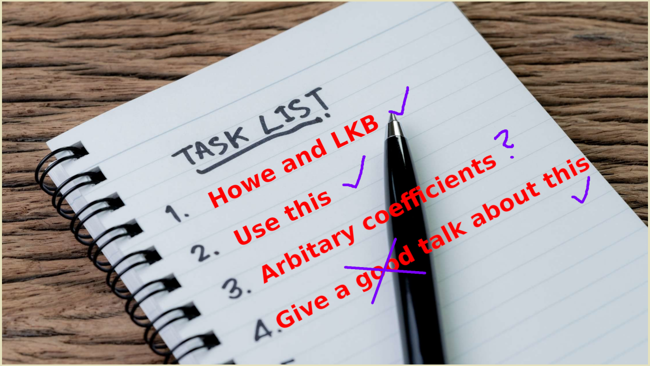
We get many Br_n-reps
and they are related to GL_m-combinatorics

on

G

Tasks

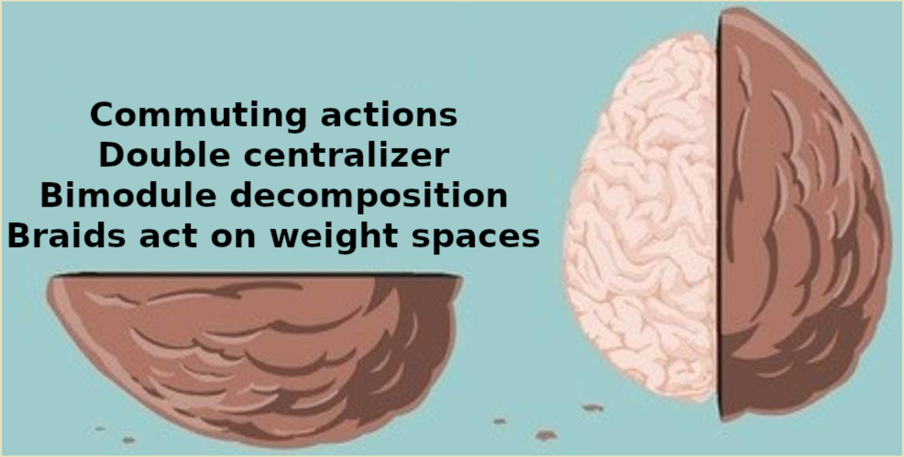
Find the LKB rep in Howe's setting
Use GL_m-combinatorics to study them



- Let us
- Examp … es an action of the

GL_n-action

Howe's approach in a nutshell



of the n-strand braid group Br_n Braid reps

Where does the LKB representation come from?

```
B = BraidGroup(3)
b = B([1])
b.LKB_matrix(variables='q,t')
[ -q^2*t          0 q^2 - q]
[      0          0       q]
[      0          1  -q + 1]
```

- The LKB rep has two variables q, t
- It is easy to guess that q is a quantum group parameter
- But what is t?

Where does the LKB representation come from?

```
B
b
b.LKB_matrix(variables='q,t')
[ -q^2*t          0  q^2 - q]
[      0          0        q]
[      0          1   -q + 1]
```

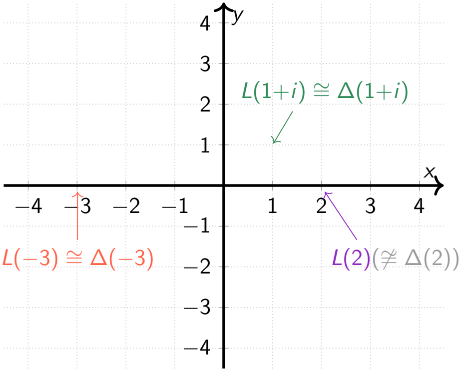
Disclaimer

The q is crucial but harmless and I will ignore it, i.e. $q = 1$!

- The LKB rep has two variables q, t
- It is easy to guess that q is a quantum group parameter
- But what is t?

Where does the LKB representation come from?

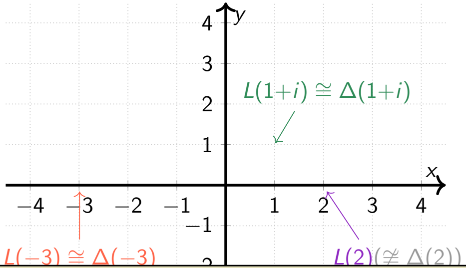
$\mathbb{C}$ = indexing set for simples



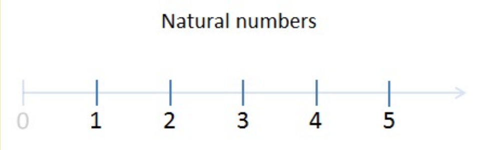
- **Verma+Bernstein–Gelfand–Gelfand ∼1966++** there exists a category $\mathcal{O}$ of $\mathfrak{sl}_2(\mathbb{C})$-modules whose simple objects $L(t)$ are indexed by $t \in \mathbb{C}$
- There is a similar statement for other semisimple Lie algebras

Where does the LKB representation come from?

$\mathbb{C}$ = indexing set for simples



Weyl (and others) ∼1930++ classified
finite dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-modules which are indexed by $\mathbb{N}_0$

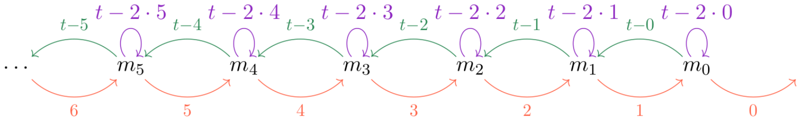


$\mathcal{O}$ fills in the remaining complex numbers

- **Verm** … ategory $\mathcal{O}$ of $\mathfrak{sl}_2$
- There is a similar statement for other semisimple Lie algebras

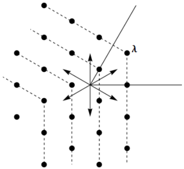
Where does the LKB representation come from?

$\mathfrak{sl}_2$:



E moves to the right, F moves to the left, H is a loop.

$\mathfrak{sl}_3$:



- For $t \notin \mathbb{N}$ the simples are so-called **Verma modules** $\Delta(t)$
- These are free modules where $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ act freely up to a highest weight condition
- $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts diagonally

Where does the LKB representation come from?

Liouville's Constant

– *The earliest transcendental number* –

$$L = \sum_{n=1}^{\infty} \frac{1}{10^{n!}} = 0.110001000\ldots$$

In[2]:=

```
1 + Pi + Pi ^ 2
```

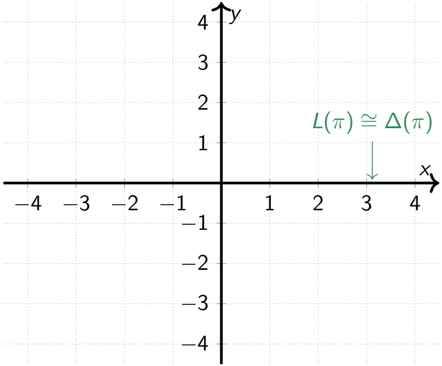
Out[2]=

$1 + \pi + \pi^2$

- **Folklore** $\sim$**???** Transcendental numbers are essentially variables
- Mathematica for example treats π as a variable unless specified otherwise
- Hence, why not take $\Delta(t)$ for t transcendental?

Where does the LKB representation come from?

$\mathbb{C}$ = indexing set for simples

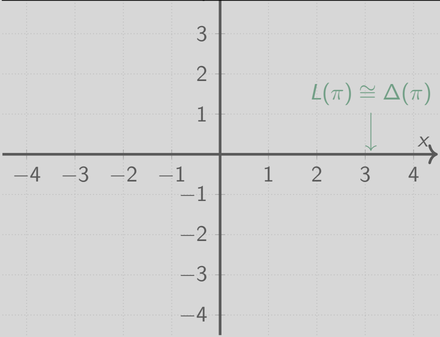


- **Jackson–Kerler ∼2009** The LKB rep of Br_n can be constructed using $\Delta(t)^{\otimes n}$ for transcendental $t \in \mathbb{C}$
- This is a Reshetikhin–Turaev type construction (different from the original)

Where does the LK... ?

Jackson–Kerler ∼2009

The construction of the LKB rep involves only R-matrices and nothing fancy!



- **Jackson–Kerler ∼2009** The LKB rep of Br_n can be constructed using $\Delta(t)^{\otimes n}$ for transcendental $t \in \mathbb{C}$
- This is a Reshetikhin–Turaev type construction (different from the original)

Where does the LK…

Jackson–Kerler $\sim$2009

The construction of the LKB rep involves only R-matrices and nothing fancy!

Kåhrström (and others) $\sim$2007

The tensor product $\Delta(t)^{\otimes n}$ naturally lives in $\tilde{\mathcal{O}}$

$\tilde{\mathcal{O}} = \mathcal{O}$ but allow countable direct sums

The tensor product is ∞ semisimple

Example $\Delta(t)^{\otimes 2} \cong \bigoplus_{s\in\mathbb{N}} \Delta(2t-2s)$

- **Jackson–Kerler $\sim$2009** The LKB rep of Br_n can be constructed using $\Delta(t)^{\otimes n}$ for transcendental $t \in \mathbb{C}$
- This is a Reshetikhin–Turaev type construction (different from the original)

Jackson–Kerler ∼2009

The construction of the LKB rep involves only R-matrices and nothing fancy!

3

Kåhrström (and others) ∼2007

The tensor product $\Delta(t)^{\otimes n}$ naturally lives in $\tilde{\mathcal{O}}$

$\tilde{\mathcal{O}} = \mathcal{O}$ but allow countable direct sums

The tensor product is ∞ semisimple

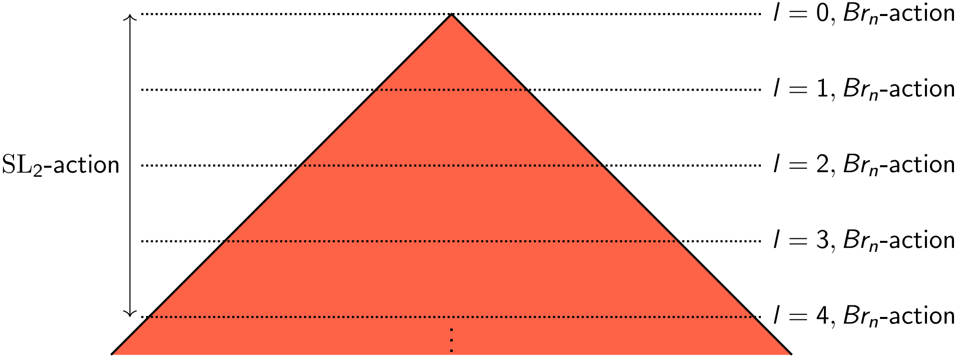
Example $\Delta(t)^{\otimes 2} \cong \bigoplus_{s\in\mathbb{N}} \Delta(2t-2s)$

-4

Ignore the tilde:
the point is that the LKB rep
comes from semisimple classical Lie theory
(plus very standard quantization)

- **Jackson–Kerle**[r] … [con]structed using $\Delta(t)^{\otimes n}$ for tr…
- This is a Reshetikhin–Turaev type construction (different from the original)

Where does the LKB representation come from?



- **Jackson–Kerler** $\sim$**2009** The ($l \in \mathbb{N}$)th LKB rep $LKB^{n,l}$ of Br_n is

$$LKB^{n,l} = \ker(E) \cap \ker\big(H - (nt - 2l)\big)$$

- Examples $l = 0 \leftrightsquigarrow$ trivial, $l = 1 \leftrightsquigarrow$ red. Burau, $l = 2 \leftrightsquigarrow$ LKB

Where does the

SL$_2$-action

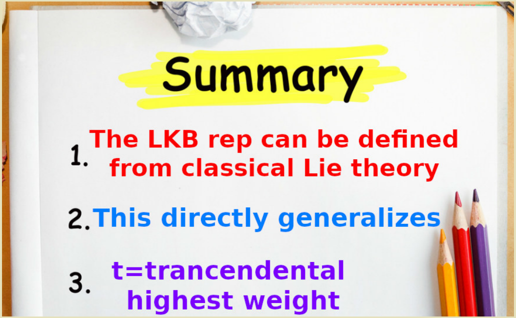
$l = 0$, Br_n-action

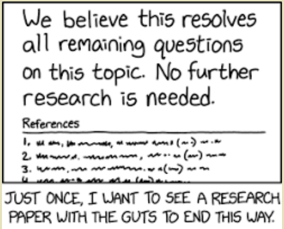
$l = 1$, Br_n-action

$l = 2$, Br_n-action

$l = 3$, Br_n-action

$l = 4$, Br_n-action





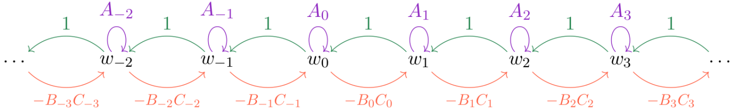
Well, maybe there is more to be said here

For example, the cone clearly suggests a Howe duality

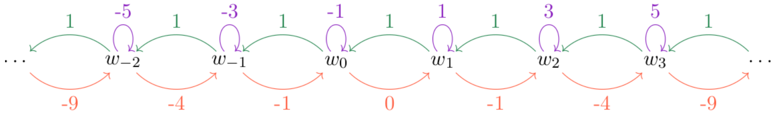
generally new, not tilting invariant and knows all of LKB

- **Jackson–** ... is
- Examples ... KB

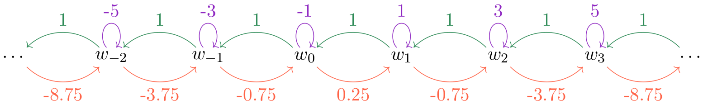
Verma Howe duality

$\mathfrak{sl}_2$:


E moves to the right, F moves to the left, K is a loop.

explicit:


E moves to the right, F moves to the left, K is a loop.

explicit:


E moves to the right, F moves to the left, K is a loop.

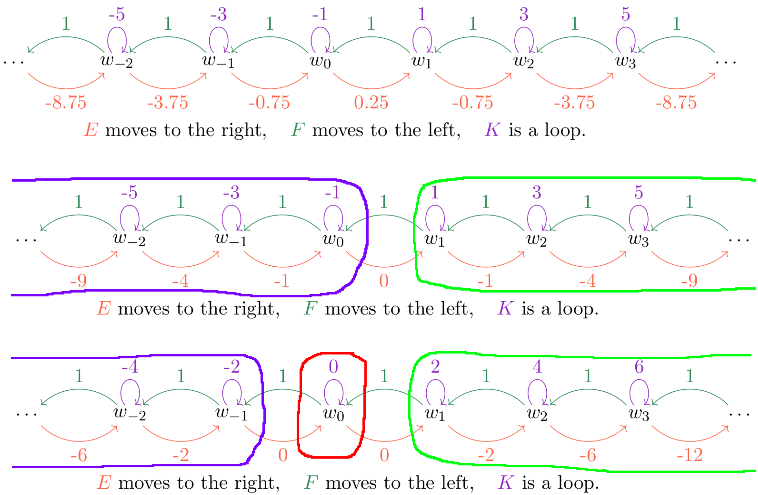
- Dense modules of $\mathfrak{sl}_n(\mathbb{C})$ = weight + support equals a coset from $\mathfrak{h}^*/Q$
- The above are examples of dense $\mathfrak{sl}_2(\mathbb{C})$-module

Verma Howe duality

simple:

coVerma / Verma:

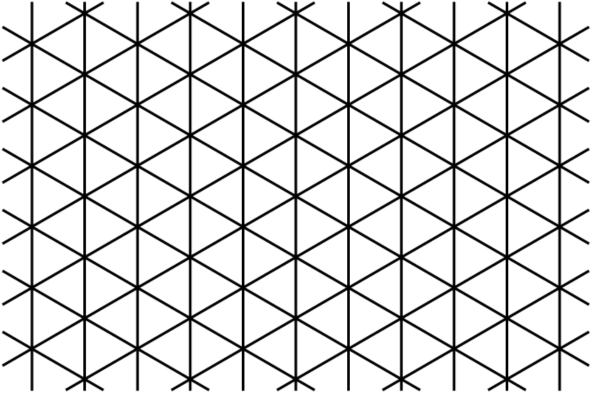
coVerma / findim / Verma:



- For $\mathfrak{sl}_2(\mathbb{C})$ these can be classified: we only have three classes on $\mathbb{Z}$

Verma Howe duality

$\mathfrak{sl}_3$:



- For higher rank these are grid-type modules; generic scalars $\Rightarrow$ simple
- **Fernando, Futorny ∼1986** All simple weight modules are dense or are induced from dense modules

Ver

Dense modules = "biggest possible" weight modules

Although the simple dense modules are essentially classified **(Mathieu ∼2000)** they still are mostly mysterious

induced from dense modules

Verma Howe duality

(a) *There are commuting actions* **Commuting actions**

$$U(\mathfrak{gl}_2) \circlearrowright \Delta^{\oplus\boldsymbol{\lambda}} = \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n} \Delta^{\lambda_1+d_1} \otimes \dots \otimes \Delta^{\lambda_n+d_n} \circlearrowleft U(\mathfrak{gl}_n).$$

(b) *Let ϕ^k be the algebra homomorphism induced by the $U(\mathfrak{gl}_k)$ actions from (a). Then, for admissible parameters $\boldsymbol{\lambda}$:* **Double centralizer**

$$\phi^2 \colon U(\mathfrak{gl}_2) \twoheadrightarrow_d \mathrm{End}_{U(\mathfrak{gl}_n)}(\Delta^{\oplus\boldsymbol{\lambda}}), \quad \phi^n \colon U(\mathfrak{gl}_n) \twoheadrightarrow_d \mathrm{End}_{U(\mathfrak{gl}_2)}(\Delta^{\oplus\boldsymbol{\lambda}}).$$

That is, the two actions densely-generate the others centralizer.

(c) *For admissible parameters $\boldsymbol{\lambda}$ we have the decomposition of the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule $\Delta^{\boldsymbol{\lambda}}$ into* **Bimodule decomp.**

$$\Delta^{\oplus\boldsymbol{\lambda}} \cong \bigoplus_{\substack{g\in\mathbb{Z}\\ t\in\mathbb{Z}_{\geq 0}}} \Delta^{\Sigma\lambda_n+g-t,t} \otimes \mathtt{D}^{g-t,t}.$$

Verma Dense

The various $\Delta^{\Sigma\lambda_n+g-t,t}$ and $\mathtt{D}^{g-t,t}$ are nonisomorphic simple $U(\mathfrak{gl}_2)$ modules respectively $U(\mathfrak{gl}_n)$ modules.

- The above is Verma Howe duality
- Admissible parameters $\Leftrightarrow$ generic/transcendental

Verma Howe duality

(a) *There are commuting actions*

$$U(\mathfrak{gl}_2) \circlearrowright \Delta^{\oplus\boldsymbol{\lambda}} = \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n} \Delta^{\lambda_1+d_1} \otimes \ldots \otimes \Delta^{\lambda_n+d_n} \circlearrowleft U(\mathfrak{gl}_n).$$

Braid action

(b) *Let ϕ^k be the algebra homomorphism induced by the $U(\mathfrak{gl}_k)$ actions from (a). Then, for admissible parameters $\boldsymbol{\lambda}$:*

$$\phi^2 \colon U(\mathfrak{gl}_2) \twoheadrightarrow_d \mathrm{End}_{U(\mathfrak{gl}_n)}(\Delta^{\oplus\boldsymbol{\lambda}}), \quad \phi^n \colon U(\mathfrak{gl}_n) \twoheadrightarrow_d \mathrm{End}_{U(\mathfrak{gl}_2)}(\Delta^{\oplus\boldsymbol{\lambda}}).$$

That is, the two actions densely-generate the others centralizer.

(c) *For admissible parameters $\boldsymbol{\lambda}$ we have the decomposition of the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule $\Delta^{\boldsymbol{\lambda}}$ into*

$$\Delta^{\oplus\boldsymbol{\lambda}} \cong \bigoplus_{\substack{g\in\mathbb{Z} \\ t\in\mathbb{Z}_{\geq 0}}} \Delta^{\Sigma\lambda_n+g-t,t} \otimes \mathtt{D}^{g-t,t}.$$

The various $\Delta^{\Sigma\lambda_n+g-t,t}$ and $\mathtt{D}^{g-t,t}$ are nonisomorphic simple $U(\mathfrak{gl}_2)$ modules respectively $U(\mathfrak{gl}_n)$ modules.

- Moreover, there is a surjective braid group action on the marked part
- All higher LKB reps shows up infinitely many times
- This "immediately" implies that all higher LKB reps are simple

Verma Howe duality

(a) *There are commuting actions*

$$U(\mathfrak{gl}_2) \circlearrowright \Delta^{\oplus\boldsymbol{\lambda}} = \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^n} \Delta^{\lambda_1+d_1} \otimes \ldots \otimes \Delta^{\lambda_n+d_n} \circlearrowleft U(\mathfrak{gl}_n).$$

(b) *Let ϕ^k be the algebra homomorphism induced by the $U(\mathfrak{gl}_k)$ actions from (a). Then, for admissible parameters $\boldsymbol{\lambda}$:*

$$\phi^2 \colon U(\mathfrak{gl}_2) \twoheadrightarrow_d \mathrm{End}_{U(\mathfrak{gl}_n)}(\Delta^{\oplus\boldsymbol{\lambda}}), \quad \phi^n \colon U(\mathfrak{gl}_n) \twoheadrightarrow_d \mathrm{End}_{U(\mathfrak{gl}_2)}(\Delta^{\oplus\boldsymbol{\lambda}}).$$

That is, the two actions densely-generate the others centralizer.

(c) *For admissible parameters $\boldsymbol{\lambda}$ we have the decomposition of the $U(\mathfrak{gl}_2)$-$U(\mathfrak{gl}_n)$ bimodule $\Delta^{\boldsymbol{\lambda}}$ into*

$$\Delta^{\oplus\boldsymbol{\lambda}} \cong \bigoplus_{\substack{g\in\mathbb{Z} \\ t\in\mathbb{Z}_{\geq 0}}} \Delta^{\Sigma\lambda_n+g-t,t} \otimes \mathtt{D}^{g-t,t}.$$

The various $\Delta^{\Sigma\lambda_n+g-t,t}$ and $\mathtt{D}^{g-t,t}$ are nonisomorphic simple $U(\mathfrak{gl}_2)$ modules respectively $U(\mathfrak{gl}_n)$ modules.

- There is also a quantum version — Easy
- There is also a higher rank version — Difficult (not done)
- There is also a nonsemisimple version — Very difficult (not done)

Ve

(a)

(b)

(c)

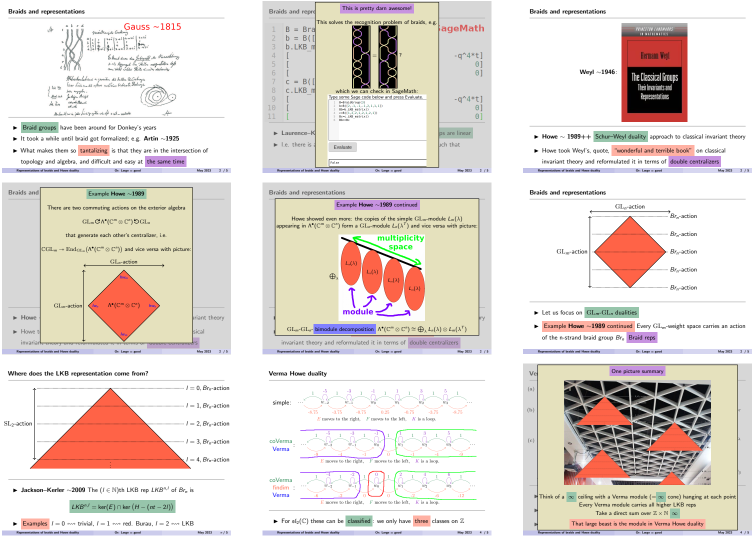
One picture summary

Think of a ∞ ceiling with a Verma module ($=\infty$ cone) hanging at each point

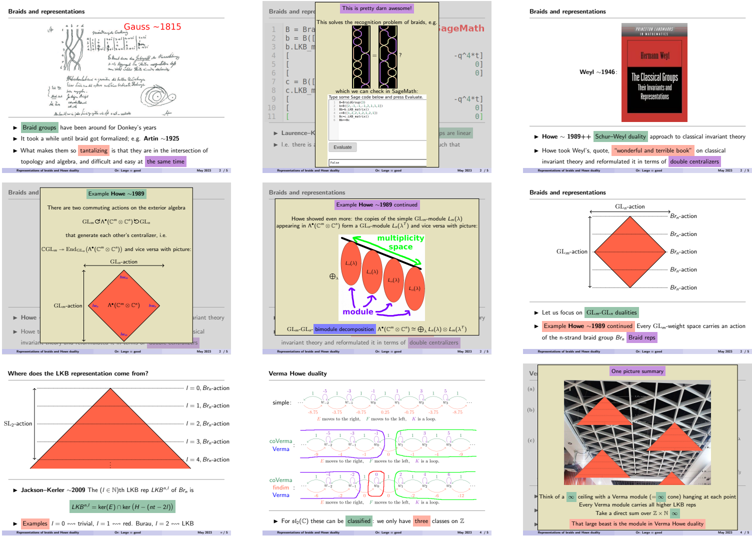
Every Verma module carries all higher LKB reps

Take a direct sum over $\mathbb{Z}\times\mathbb{N}$ ∞

That large beast is the module in Verma Howe duality



There is still much to do...



Thanks for your attention!