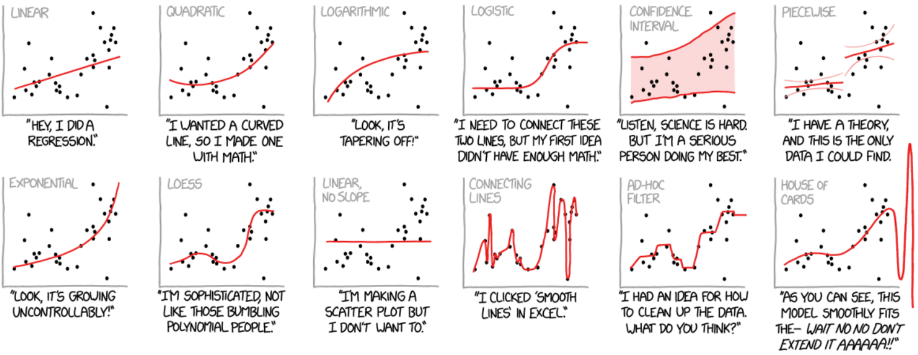
Examples of analytic methods in tensor categories

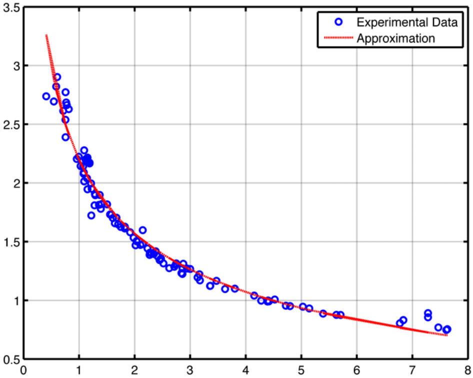
Or: Assume n is very large



I report on work of Kevin Coulembier, Pavel Etingof and Victor Ostrik

April 2023

Let us not count!



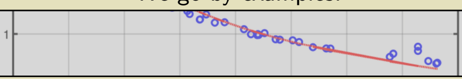
- Observation Many problems are only difficult because we like exact solutions
- Bonus observation Many difficult problems are easy for large subclasses
- Analytic method **(Folklore ∼very early)** Approximate answers are often much easier to get

Let us not cou...

If you do not know what this means in general

you are in ~~good~~some company: I do not know either!

We go by examples!



Example/appetizer 1 from number theory – historically the first of its kind!?

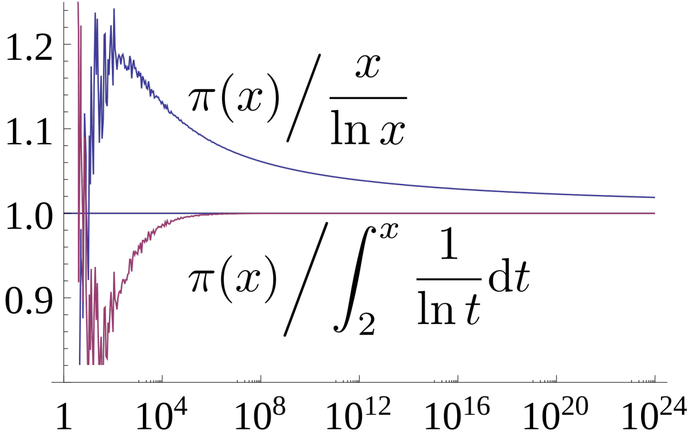
Example/appetizer 2 from graph theory – easy to understand and prototypical

Example 3 from representation theory – prototypical

Example 4 from tensor categories – finally there

- ...ions
- ...
- Analytic method ... Approximate answers are often much easier to get

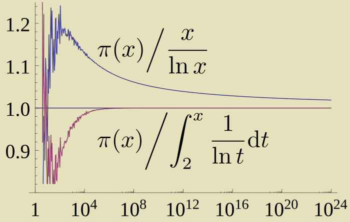
Let us not count!

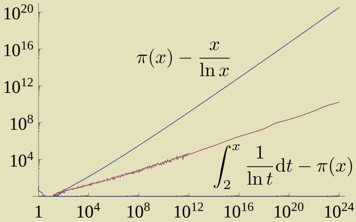


- Counting primes is difficult but...
- **Prime number theorem (many people ∼1793)** #primes $= \pi(n) \sim n / \ln n$

Let us ...

$\sim$ means asymptotically = ratios are good (not the absolute difference!)





So this is not doing the count!

- C...
- **Prime number theorem (many people ~1793)** #primes $= \pi(n) \sim n/\ln n$

Let

Seriously, counting is difficult!

Legendre ~1808:

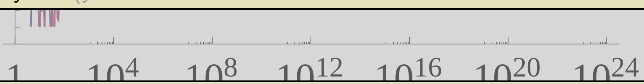
(for $n/(\ln n - 1.08366)$)

Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

Acctually, #primes<1000 =1229...

Gauss, Legendre and company counted primes up to $n = 400000$ and more

That took years (your IPhone can do that in seconds...humans have advanced!)



The prime number theorem gave birth to **analytic number theory**

Analytic number theory is full of
"discrete statements solved approximately"

- Coun
- **Prime number theorem (many people ~1793)** #primes $= \pi(n) \sim n/\ln n$

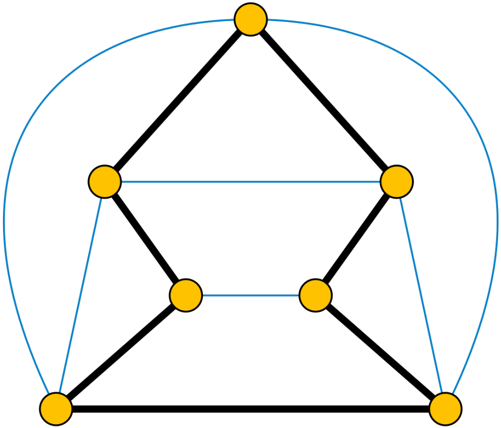
Let us not count!

A very difficult problem might have a nice answer

if we are happy with nonexact solutions

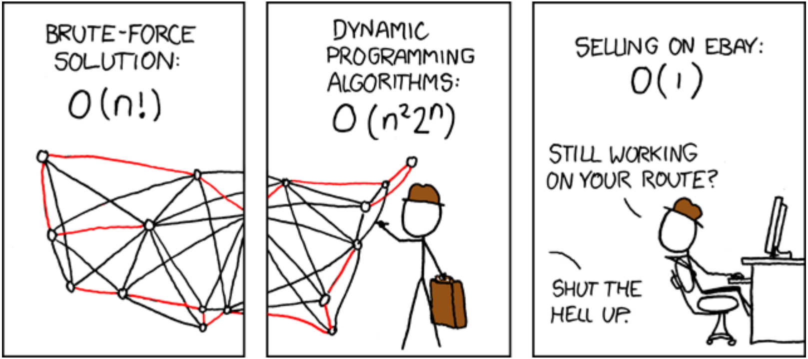
- Counting primes is difficult but...
- **Prime number theorem (many people ∼1793)** #primes $= \pi(n) \sim n/\ln n$

Let us not count!



- Hamiltonian cycle = a cycle that visits every vertex exactly once
- Hamiltonian graph = a graph with an Hamiltonian cycle

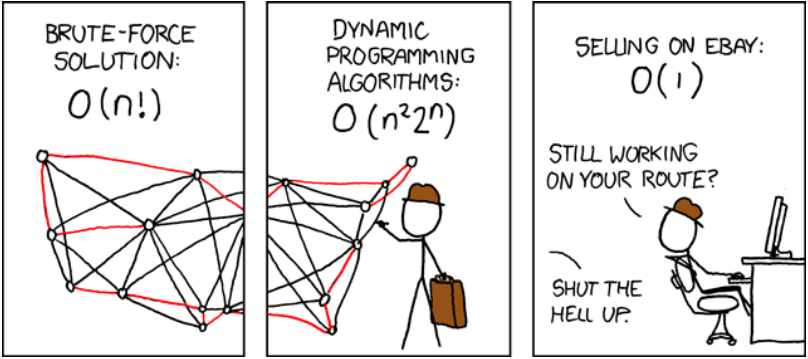
Let us not count!



(This is the traveling salesperson problem.)

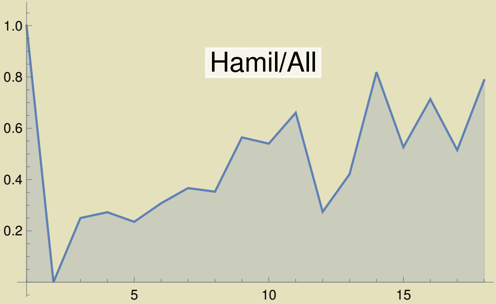
- Hamiltonian graph was one of the first problems shown to be NP-complete
- NP-complete "=" can't do much better than brute force
- Dynamic programming algorithms solves this is roughly in $O(n^2 2^n)$, $n = \#V$

Let us not count!



- To determine precisely whether a graph is Hamiltonian is difficult
- To determine approximately whether a graph is Hamiltonian is easy
- **Pósa∼1976** Choosing a graph randomly, the probability is 1 that the graph is Hamiltonian: $\lim_{n\to\infty} P(Hamil) = 1$ (probability)

Let u



The precise statement

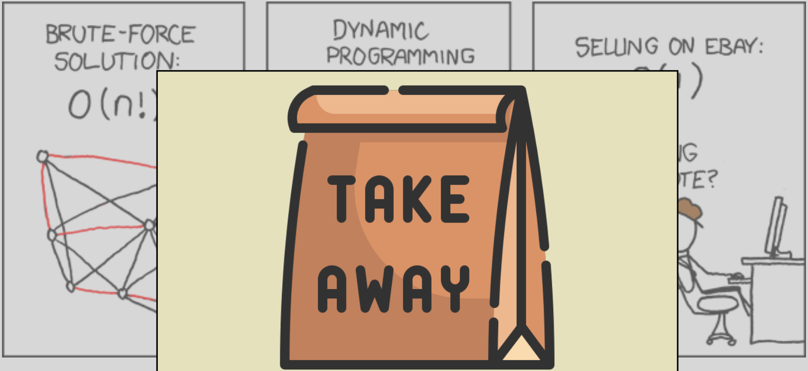
Pósa showed that $P(Hamil) \to 1$ for $n \to \infty$ and graphs with $cn \ln n$ edges
This implies #HamilGraphs on $\leq n$ vertices / #AllGraphs on $\leq n$ vertices
goes to zero for $n \to \infty$ since most graphs have $\geq c'n^2$ edges

The proof of this theorem
is a not difficult counting argument

In general, graph theory
provides many statements of the form
"XYZ is very difficult, but we can solve it approximately"

- To determ ... ult
- To determ ... easy
- **Pósa∼19** ... hat the graph is Hamiltonian: $\lim_{n\to\infty} P(Hamil) = 1$ (probability)

Let us not count!



A very difficult problem might have a nice answer

almost all of the time

- To determine … difficult
- To determine … n is easy
- **Pósa∼1976** Choosing a graph randomly, the probability is 1 that the graph is Hamiltonian: $\lim_{n\to\infty} P(Hamil) = 1$ (probability)

What about representation theory?

char table of M_{11}:

Class		1	2	3	4	5	6	7	8	9	10
Size		1	165	440	990	1584	1320	990	990	720	720
Order		1	2	3	4	5	6	8	8	11	11
p = 2		1	1	3	2	5	3	4	4	10	9
p = 3		1	2	1	4	5	2	7	8	9	10
p = 5		1	2	3	4	1	6	8	7	9	10
p = 11		1	2	3	4	5	6	7	8	1	1
X.1	+	1	1	1	1	1	1	1	1	1	1
X.2	+	10	2	1	2	0	-1	0	0	-1	-1
X.3	0	10	-2	1	0	0	1	Z1	-Z1	-1	-1
X.4	0	10	-2	1	0	0	1	-Z1	Z1	-1	-1
X.5	+	11	3	2	-1	1	0	-1	-1	0	0
X.6	0	16	0	-2	0	1	0	0	0	Z2	Z2#2
X.7	0	16	0	-2	0	1	0	0	0	Z2#2	Z2
X.8	+	44	4	-1	0	-1	1	0	0	0	0
X.9	+	45	-3	0	1	0	0	-1	-1	1	1
X.10	+	55	-1	1	-1	0	-1	1	1	0	0

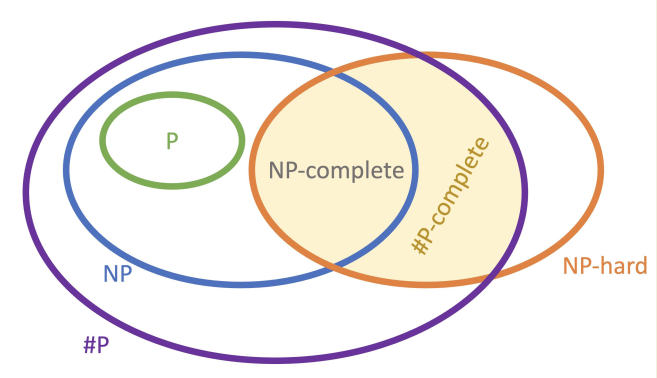
- We now discuss finite groups G with fd reps over $\mathbb{C}$
- **Burnside $\sim$1911** Every >1d simple character has zeros
- Question Determine where the zeros are

What about representation theory?

A problem we have seen before :

Frobenius ∼1895++ Character formulas for S_n

Hepler ∼1994 (potentially known earlier)
To determine the zeros for S_n is #P complete (=very difficult)



ch

- We no
- **Burns**
- Question Determine where the zeros are

What about representation theory?

char table of S_4:

```
-----------------------
Class |    1  2  3  4  5
Size  |    1  3  6  8  6
Order |    1  2  2  3  4
-----------------------
p  =  2    1  1  1  4  2
p  =  3    1  2  3  1  5
-----------------------
X.1   +    1  1  1  1  1
X.2   +    1  1 -1  1 -1
X.3   +    2  2  0 -1  0
X.4   +    3 -1 -1  0  1
X.5   +    3 -1  1  0 -1
```

$$P(\chi(g)=0)=24/120\approx 0.194, \quad P(\chi(C)=0)=4/25=0.16$$

- **Problem** Determine for which $g\in G$ we have $\chi(g)=0$ **Too hard!**
- **Better(?) problem** $P(\chi(g)=0)$ or $P(\chi(C)=0)$ (probability) for randomly chosen $g\in G$ or conjugacy class C

What about representation theory?

char table of S_7:

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Class		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Size		1	21	105	105	70	280	210	630	504	210	420	840	720	504	420
Order		1	2	2	2	3	3	4	4	5	6	6	6	7	10	12
p = 2		1	1	1	1	5	6	4	4	9	5	5	6	13	9	10
p = 3		1	2	3	4	1	1	7	8	9	4	2	3	13	14	7
p = 5		1	2	3	4	5	6	7	8	1	10	11	12	13	2	15
p = 7		1	2	3	4	5	6	7	8	9	10	11	12	1	14	15
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	-1	-1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1
X.3	+	6	-4	0	2	3	0	-2	0	1	-1	-1	0	-1	1	1
X.4	+	6	4	0	2	3	0	2	0	1	-1	1	0	-1	-1	-1
X.5	+	14	6	2	2	2	-1	0	0	-1	2	0	-1	0	1	0
X.6	+	14	-6	-2	2	2	-1	0	0	-1	2	0	1	0	-1	0
X.7	+	14	-4	0	2	-1	2	2	0	-1	-1	-1	0	0	1	-1
X.8	+	14	4	0	2	-1	2	-2	0	-1	-1	1	0	0	-1	1
X.9	+	15	5	-3	-1	3	0	1	-1	0	-1	-1	0	1	0	1
X.10	+	15	-5	3	-1	3	0	-1	-1	0	-1	1	0	1	0	-1
X.11	+	20	0	0	-4	2	2	0	0	0	2	0	0	-1	0	0
X.12	+	21	1	-3	1	-3	0	-1	-1	1	1	1	0	0	1	-1
X.13	+	21	-1	3	1	-3	0	1	-1	1	1	-1	0	0	-1	1
X.14	+	35	-5	-1	-1	-1	-1	1	1	0	-1	1	-1	0	0	1
X.15	+	35	5	1	-1	-1	-1	-1	1	0	-1	-1	1	0	0	-1

$$P(\chi(g)=0)=28146/75600\approx 0.372,\quad P(\chi(C)=0)=55/225\approx 0.24$$

- **Problem** Determine for which $g\in G$ we have $\chi(g)=0$ **Too hard!**
- **Better(?) problem** $P(\chi(g)=0)$ or $P(\chi(C)=0)$ for randomly chosen $g\in G$ or conjugacy class C

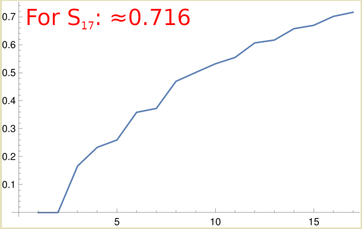
What about representa…

char table of …

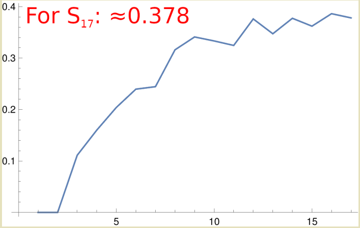
12	13	14	15
840	720	504	420
6	7	10	12
6	13	9	10
3	13	14	7
12	13	2	15
12	1	14	15
1	1	1	1
-1	1	-1	-1
0	-1	1	1
0	-1	-1	-1
-1	0	1	0
1	0	-1	0
0	0	1	-1
0	0	-1	1
0	1	0	1
0	1	0	-1
0	-1	0	0
0	0	1	-1
0	0	-1	1
-1	0	0	1
1	0	0	-1

X.5 + 14 6 2 2 2 -1 0 0 -1 2 0

Here is $P(\chi(g)=0)$:



Here is $P(\chi(C)=0)$:



$P(\chi(g)=0)=281$ … $0)=55/225\approx 0.24$

- P…
- B… $g\in G$

or

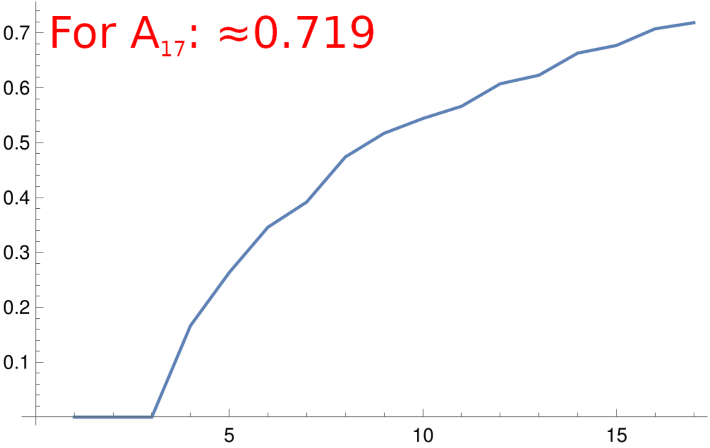
My silly 1-hour-work code only made it to S_{17}, pathetic, sorry for that!

Alexander Miller computed these up to S_{38}

Anyway, we can guess from here!

What about representation theory?

for up to A_{17}:



- **Miller** $\sim$**2013** Choosing S_n, $g \in S_n$ and χ simple character of S_n randomly, the probability is 1 that $\chi(g) = 0$ (formally, $\lim_{n\to\infty} P(\chi(g) = 0) = 1$)
- $\lim_{n\to\infty} P(\chi(C) = 0) =?$, but this is likely neither 0 nor 1 !

Many people ∼1911++ There are many groups with the same type of behavior and also many related statements

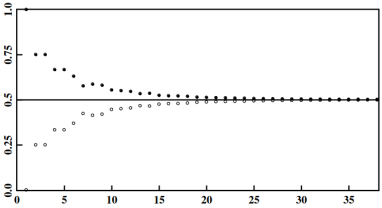


FIGURE 2. The plot • for Prob($\chi(\mu) > 0 \mid \chi(\mu) \neq 0$) and the plot ∘ for Prob($\chi(\mu) < 0 \mid \chi(\mu) \neq 0$) where $1 \leq n \leq 38$.

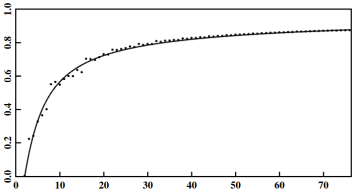


FIGURE 3. The proportion of the character table of S_n covered by even entries for $2 \leq n \leq 76$ and the graph of $2\pi^{-1} \arctan(\sqrt{n/2} - 1)$ for $2 \leq n \leq 76$.

These tables are due to Alexander Miller

What about representation theory?

For A_{17}: ≈ 0.719

for up to

A very difficult problem might have a nice answer

almost all of the time – similarly to the Hamiltonian graph problem

- **Mill** ndomly, the probability is 1 that $\chi(g) = 0$ (formally, $\lim_{n\to\infty} P(\chi(g) = 0) = 1$)
- $\lim_{n\to\infty} P(\chi(C) = 0) = ?$, but this is likely neither 0 nor 1 !

What about representation theory?

- Γ = any affine semigroup superscheme, $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Γ has the notion of a tensor product
- Problem Decompose $V^{\otimes n}$; note that $\dim V^{\otimes n} = (\dim V)^n$

What about representation theory?

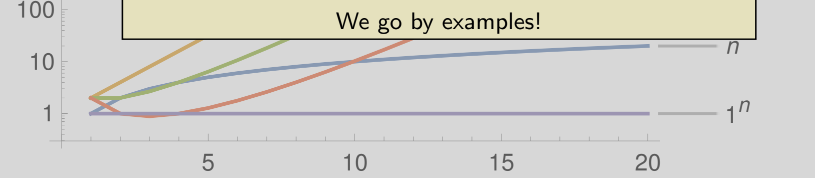
dim $V = 1$ works perfectly well
but then my story about **exponential growth** is flawed
so I ignore dim $V = 1$ and assume dim $V > 1$

- Γ = any affine semigroup superscheme, $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Γ has the notion of a tensor product
- **Problem** Decompose $V^{\otimes n}$; note that **$\dim V^{\otimes n} = (\dim V)^n$**

What about

If you do not know what an affine semigroup superscheme is you are in ~~good~~some company: **I do not know either!**

We go by examples!



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- **Problem** Decompose $V^{\otimes n}$; note that $\dim V^{\otimes n} = (\dim V)^n$

What about

10^6 10^5 10^4 1000 100

m $V)^n$

m $V)^n$ / n

m $V)^n$ / n^2

If you do not know what an affine semigroup superscheme is you are in ~~good~~some company: I do not know either!

We go by examples!

Examples

Any finite group, monoid, semigroup

Symmetric groups, alternating groups, cyclic groups, the monster, $GL_N(\mathbb{F}_{p^k})$, ...

Actually any group, monoid, semigroup

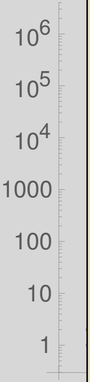
$GL_N(\mathbb{C})$, $GL_N(\mathbb{R})$, $GL_N(\overline{\mathbb{F}_{p^k}})$, symplectic, orthogonal, braid groups, Thompson groups, ...

Super versions

$GL_{M|N}$, $OSP_{M|2N}$, periplectic, queer, ...

Slogan This is a very general setting

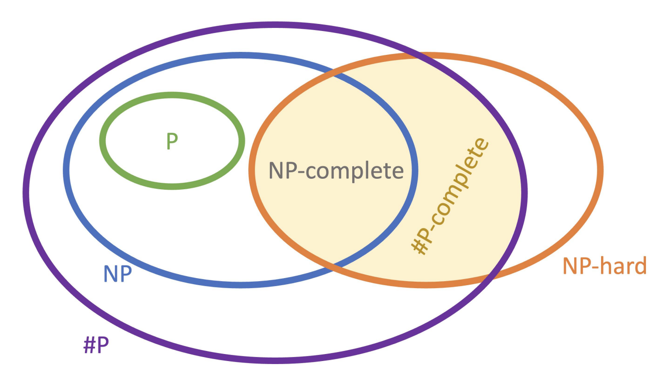
What ab



A problem we have seen before:

Murnaghan ∼1938 Asked to decompose $V^{\otimes n}$ over $\mathbb{C}$ for S_n and V simple (Kronecker coefficients)

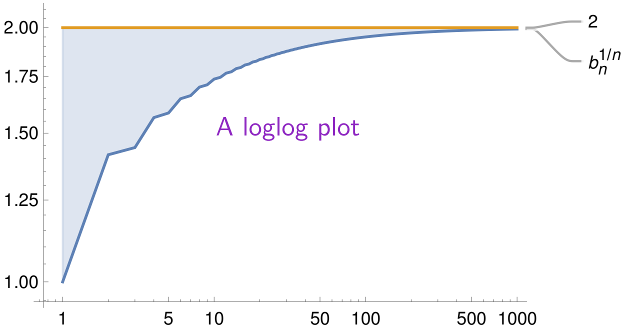
Hepler ∼1994 (potentially known earlier)
Computing Kronecker coefficients is #P complete (=very difficult)



This is a very special case of what CEO want to do...

- Γ = a ... Γ-rep
- Γ has
- Prob

What about representation theory?

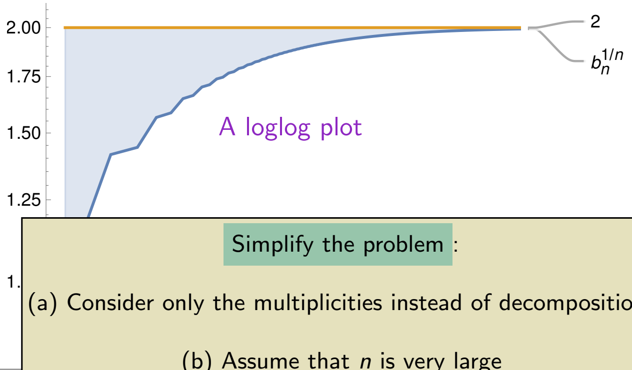


- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, \dots, 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $2 = \dim V$: $\sqrt[1000]{b_{1000}} \approx 1.99265$

What about representation theory?



Simplify the problem:

(a) Consider only the multiplicities instead of decompositions

(b) Assume that n is very large

- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

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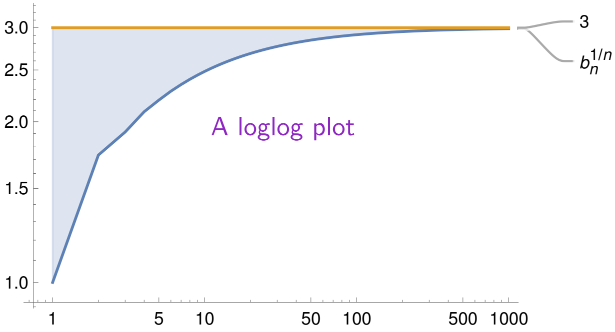

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- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathrm{Sym}\,\mathbb{C}^2$, then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, ..., 10.$$

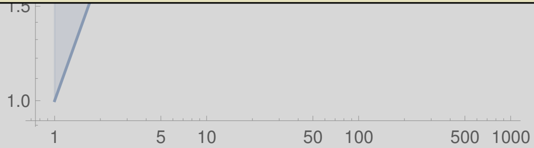
$\lim_{n\to\infty} b_n$ seems to converge to $3 = \dim V$: $\sqrt[1000]{b_{1000}} \approx 2.9875$

Wha

Observation 1

Whatever is true for SL_2 over $\mathbb{C}$ is true in general, right?

So let us come back to the general setting:
Γ = affine semigroup superscheme
$\mathbb{K}$ = any field, V = any fin dim Γ-rep
$b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)



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$\Gamma =$ affine semigroup superscheme
$\mathbb{K} =$ any field, $V =$ any fin dim Γ-rep
$b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)

1.5

Observation 2

1.0

$b_n b_m \leq b_{n+m} \Rightarrow$
$\beta = \lim_{n\to\infty} \sqrt[n]{b_n}$
is well-defined by a version of Fekete's Subadditive Lemma

- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathrm{Sym}\,\mathbb{C}^2$, then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} b_n$ seems to converge to $3 = \dim V$: $\sqrt[1000]{b_{1000}} \approx 2.9875$

Observation 1

Whatever is true for SL_2 over $\mathbb{C}$ is true in general, right?

So let us come back to the general setting:

Γ = affine semigroup superscheme

$\mathbb{K}$ = any field, V = any fin dim Γ-rep

$b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)

Observation 2

$$b_n b_m \leq b_{n+m} \Rightarrow$$

$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n}$$

is well-defined by a version of Fekete's Subadditive Lemma

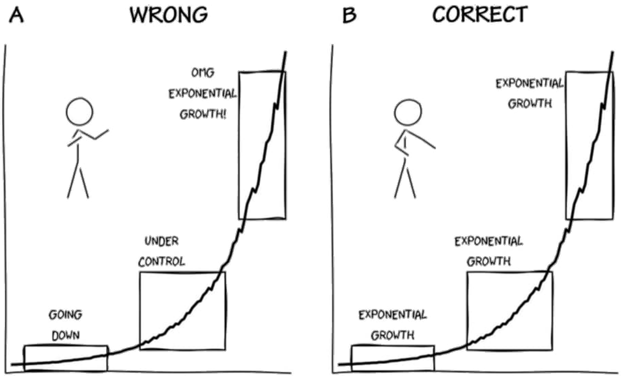
Observation 3

$$1 \leq \beta \leq \dim V$$

$\beta = 1 \Leftrightarrow V^{\otimes n}$ for $n \gg 0$ is 'one block'

$\beta = \dim V \Leftrightarrow$ summands of $V^{\otimes n}$ for $n \gg 0$ are 'essentially one-dimensional'

What about representation theory?

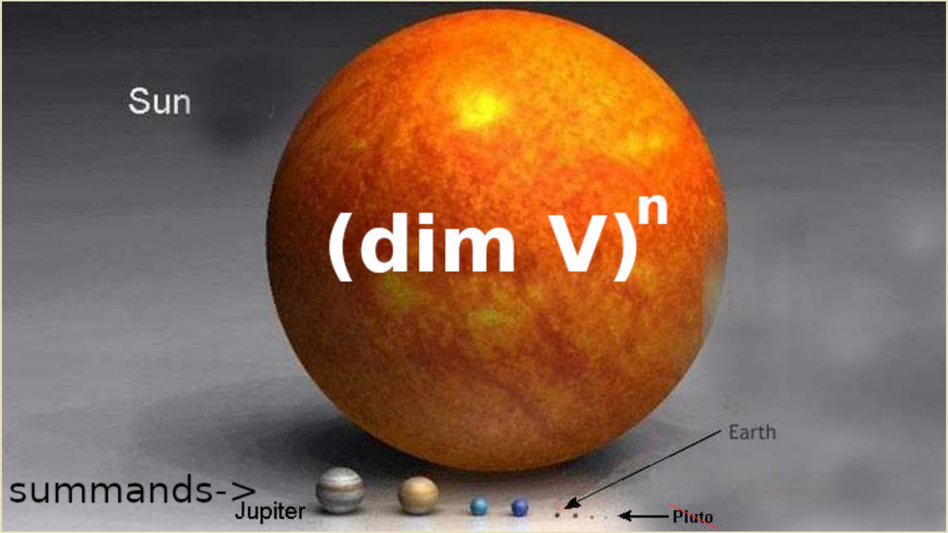


Coulembier–Etingof–Ostrik $\sim$**2023** We have

$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim V$$

Exponential growth is scary

In other words, compared to the size of the exponential growth of $(\dim V)^n$ all indecomposable summands are ‘essentially one-dimensional’



What about representation theory?

Honorable mentions

Coulembier–Etingof–Ostrik ∼2023 The same holds for any $\mathbb{K}$-linear Karoubian monoidal category that is Krull–Schmidt and has a $\mathbb{K}$-linear faithful symmetric monoidal functor to $\mathbb{K}$-vector spaces

Coulembier–Etingof–Ostrik ∼2023
Ditto in char zero when we go to super $\mathbb{K}$-vector spaces

Coulembier–Etingof–Ostrik ∼2022 Assume that our category has duals
If one only counts summands whose dim is divisible by some fixed prime then the limit is an algebraic integer in $[1, \dim V]$

Coulembier–Etingof–Ostrik ∼2023

Many more results!

Coulembier–Etingof–Ostrik ∼2023 We have

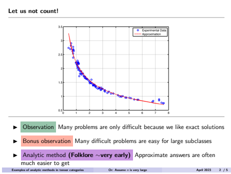
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim V$$

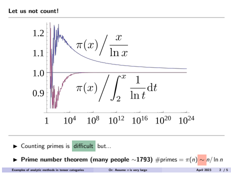
What about representation theory?

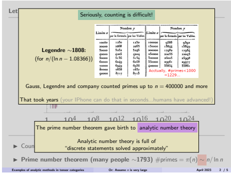
A very difficult problem might have a nice answer

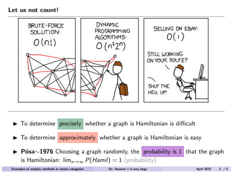
if we are happy with nonexact solutions – similarly to the prime number theorem

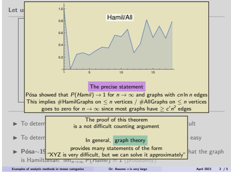
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim V$$

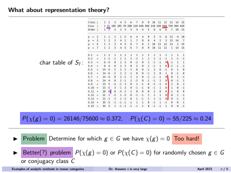


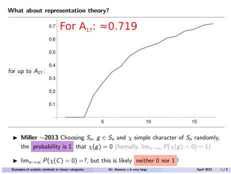


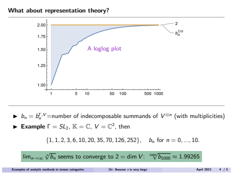


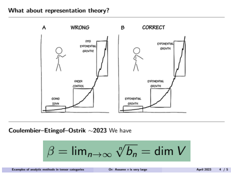




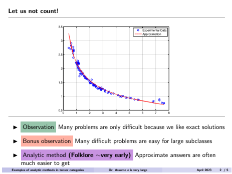


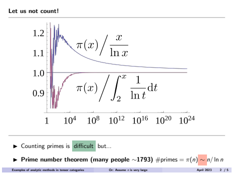


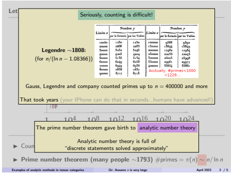


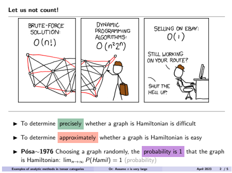


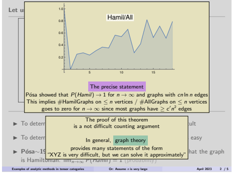
There is still much to do...

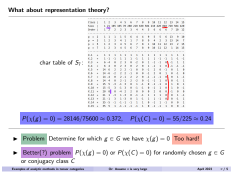


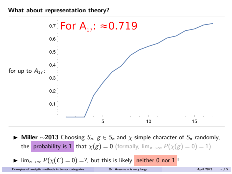


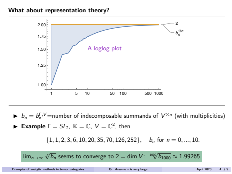


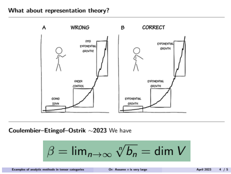












Thanks for your attention!