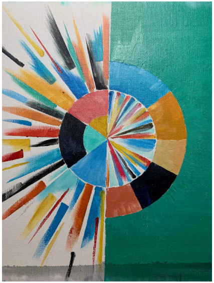
From crystals to cellularity of KLR algebras

Or: String games

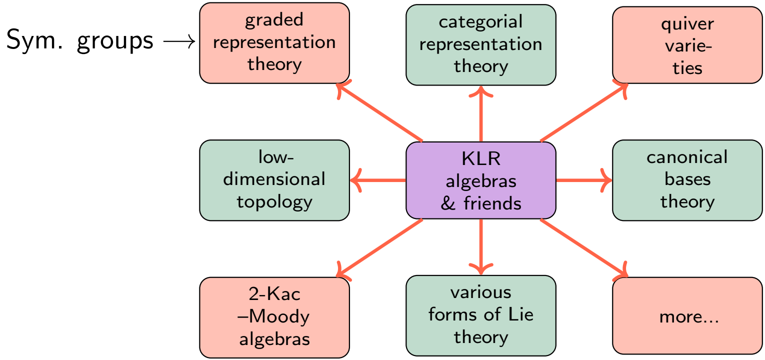
Daniel Tubbenhauer



Joint with Andrew Mathas or, honestly, I report on work of Andrew's

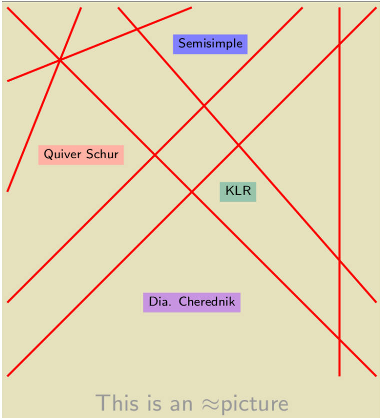
December 2022

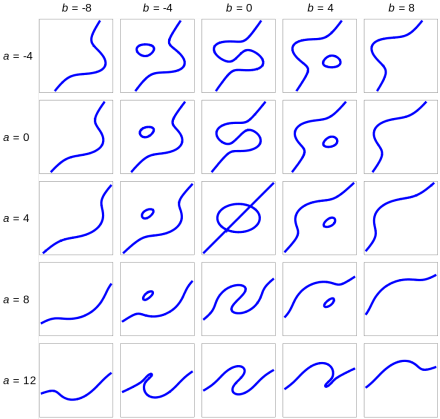
What? Why? How?



- **Khovanov–Lauda–Rouquier ~2008 + many others** (including many people here) KLR algebras are at the heart of categorical representation theory
- Problem These are actually really complicated!
- Goal Try to find nice ("cellular") bases for them

What? Why? How?

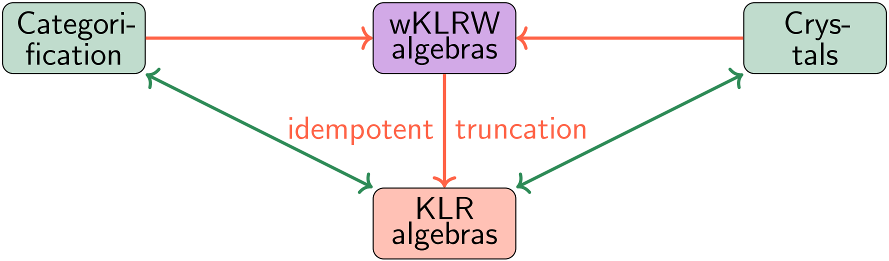




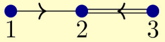
Idea **(Webster ∼2012 for KLR+friends, folklore <2012 as a general approach)**

- Use an algebra that depends on continuous parameters ⇒ wKLRW algebra
- Varying the parameters relates "important" algebras by "passing singularities"

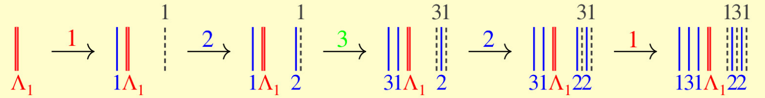
What? Why? How?



C_3 :



$$[\Lambda_1] \xrightarrow{1} [-\Lambda_1+\Lambda_2] \xrightarrow{2} [-\Lambda_2+\Lambda_3] \xrightarrow{3} [\Lambda_2-\Lambda_3] \xrightarrow{2} [\Lambda_1-\Lambda_2] \xrightarrow{1} [-\Lambda_1]$$



Use a richer combinatorics which is somewhat easier although more sophisticated

What? W

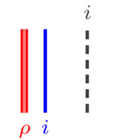
Catego ficatio

Crys- tals

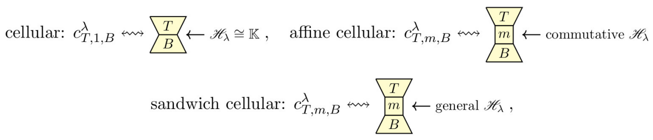
In this talk will be about,in order 3), 1), 3) and ramble on 2):

diagrammatics, cellularity and partnership:

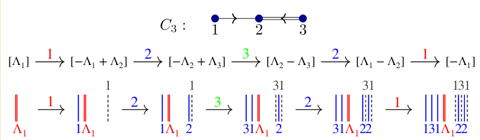
1) The diagram combinatorics



2) Sandwich cellularity



3) Idempotents and crystals partners



Use a rich ... phisticated

What? W

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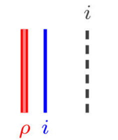
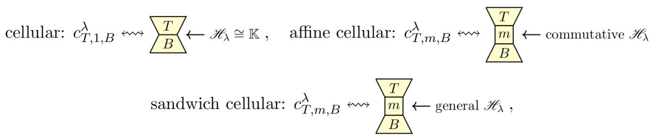


2) Sandwich cellularity

Sandwich cellularity goes back to **Brown** ∼**1955**



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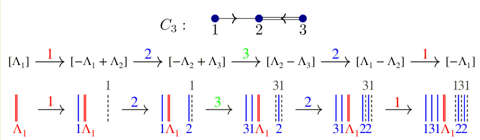
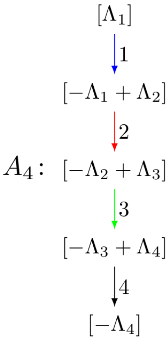


Use a rich … phisticated

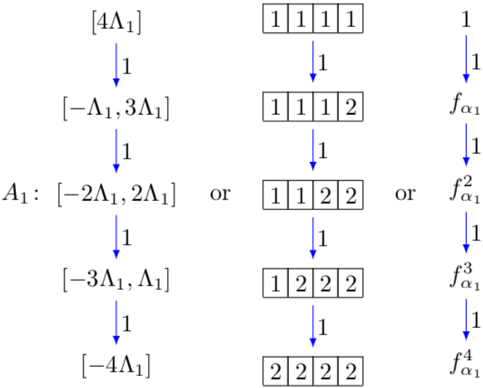
Reps at $q = 0$



- In this talk, $\mathfrak{g}$ is some Kac–Moody algebra with Chevalley generators e_i, f_i
- In essence, a crystal is a direct graph with colored edges, and it is the combinatorial shadow of a $\mathfrak{g}$-rep

vertices $\leftrightsquigarrow$ weight spaces | colored edges $\leftrightsquigarrow$ action of the f_i

Reps at $q = 0$



- **Example (above)** The simple $\mathfrak{sl}_2$-rep $Sym^4\mathbb{C}^2$ via the vanilla, tableaux, PBW flavor
- Crystal magic Get rid of all funny coefficients and summands, and only keep the "main part" of $\mathfrak{g}$-reps

Reps at $q = 0$

$$A_4\colon\ \begin{array}{c} [\Lambda_1] \\ \downarrow 1 \\ [-\Lambda_1+\Lambda_2] \\ \downarrow 2 \\ [-\Lambda_2+\Lambda_3] \\ \downarrow 3 \\ [-\Lambda_3+\Lambda_4] \\ \downarrow 4 \\ [-\Lambda_4] \end{array} \quad \text{or} \quad \begin{array}{c} \boxed{1} \\ \downarrow 1 \\ \boxed{2} \\ \downarrow 2 \\ \boxed{3} \\ \downarrow 3 \\ \boxed{4} \\ \downarrow 4 \\ \boxed{5} \end{array} \quad \text{or} \quad \begin{array}{c} 1 \\ \downarrow 1 \\ f_{\alpha_1} \\ \downarrow 2 \\ f_{\alpha_1+\alpha_2} \\ \downarrow 3 \\ f_{\alpha_1+\alpha_2+\alpha_3} \\ \downarrow 4 \\ f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \end{array}$$

- **Example (above)** The simple $\mathfrak{sl}_5$-rep $\mathbb{C}^5$ via the vanilla, tableaux, PBW flavor
- Crystal magic Get rid of all funny coefficients and summands, and only keep the "main part" of $\mathfrak{g}$-reps

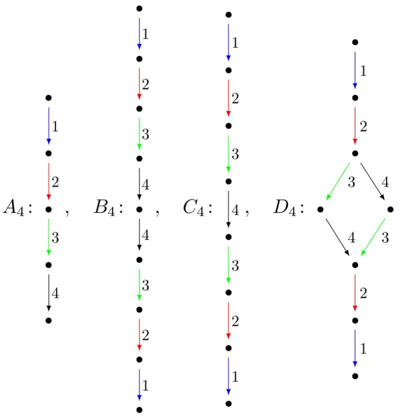
Reps at $q = 0$

$A_4 \leftrightsquigarrow SL_5(\mathbb{C})$

$B_4 \leftrightsquigarrow SO_9(\mathbb{C})$

$C_4 \leftrightsquigarrow SP_8(\mathbb{C})$

$D_4 \leftrightsquigarrow SO_8(\mathbb{C})$



- **Example (above)** The simple reps $L(\Lambda_1)$ of classical types
- Crystal magic Get rid of all funny coefficients and summands, and only keep the "main part" of $\mathfrak{g}$-reps

Reps at $q = 0$

Crystals come in many flavors:

Vanilla Works in general

Tableaux Works for classical types ABCD

PBW Works for finite types

The point Any flavor has different combinatorics

A_4

B_4

C_4

D_4

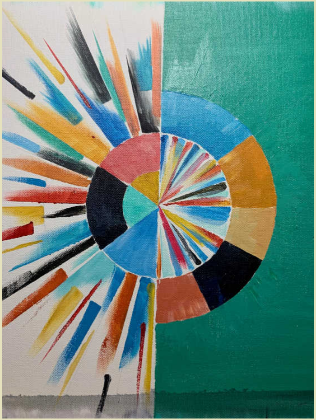
- **Example**
- Crystal magic Get rid of all funny coefficients and summands, and only keep the "main part" of $\mathfrak{g}$-reps

Reps at $q =$

Idea (Folklore $\sim$2010)

The combinatorics of crystals determines
algebraic properties of KLR/wKLRW algebras and vice versa

They might look different but are actually the "same"



A

B

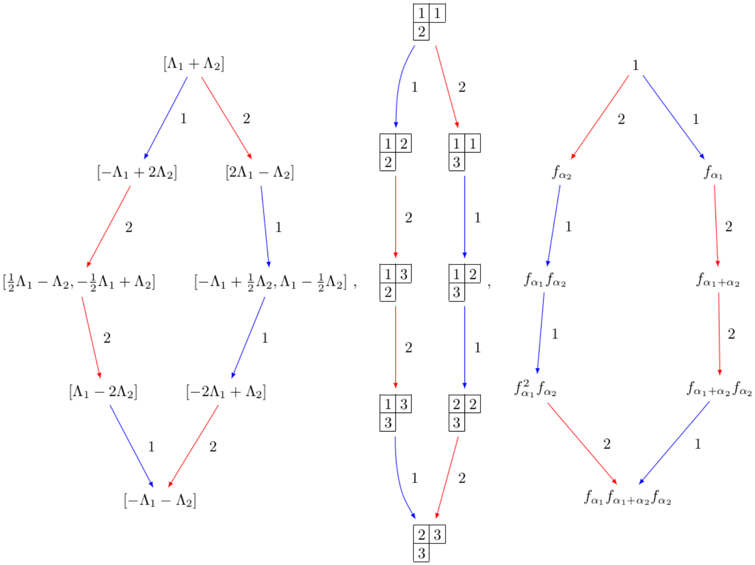
C

D

- **Exampl**
- Crystal … nd only keep the "main part" of $\mathfrak{g}$-reps

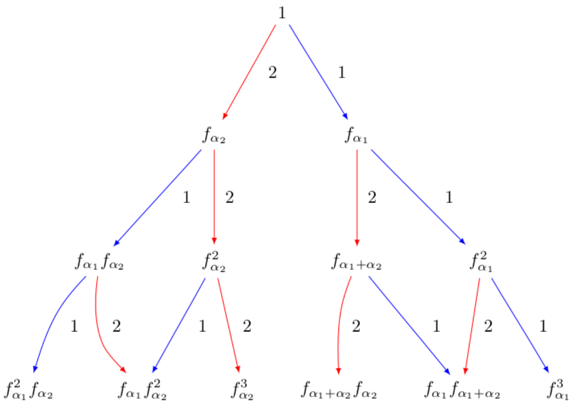
Reps at $q = 0$

Let us enjoy some crystals in type A_2:



Reps at $q = 0$

$q = 0$ of the PBW theorem:

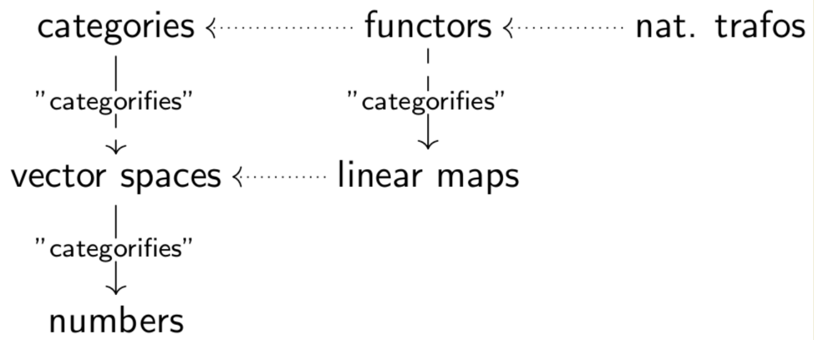


- In finite type one can cut out all crystals from a general PBW crystal
- Idea If the partnership between crystals and KLR algebras works, then finite type KLR algebra should be quite special

Reps at $q = 0$

Theorem (KLR, Brundan–Kleshchev, Kang–Kashiwara, Webster, Lauda–Vazirani, many more ∼2010)

Cyclotomic KLR algebras categorify highest weight $\mathfrak{g}$-reps
and the categories of their graded modules
have the structure of the underlying crystal



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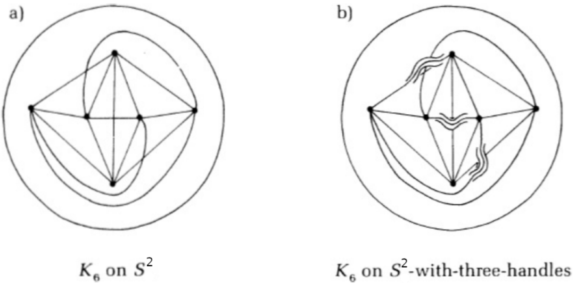


Build a bridge=handle for every crossing+ bridges get rid of intersections

- Every graph can be embedded into some orientable surface (proof above)
- Hence, we can talk about faces in crystals

Reps

Serre relations in crystals (Stembridge ∼2002, Sternberg ∼2007)

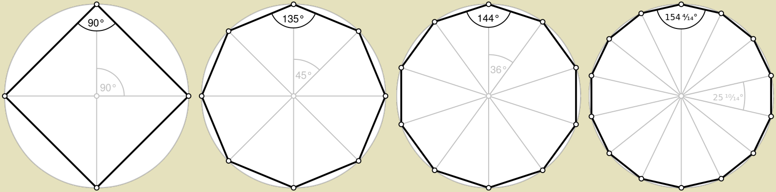
The minimal i-j faces in (fundamental) crystals of finite type are either

Tetragon=square colored $ij = ji$

Octagons=8gon colored $ijji = jiij$

Decagons=10gon colored $ijjji = jiiij = jijij$

Tetradecagon=14gon colored $ijjijij = jiijjji = jijijji = ijjjiij$



intersections

Type A has only squares

Simply-laced types have only squares and octagons

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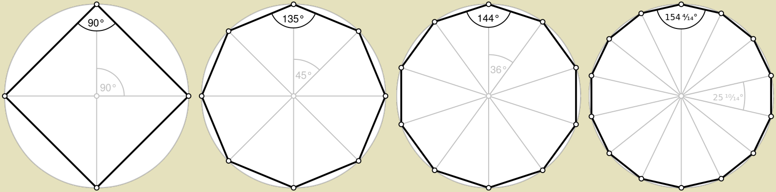
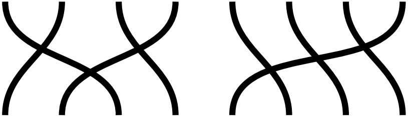


This implies for example
that there are only very! finitely many
relation to show in the KLR world
in our story

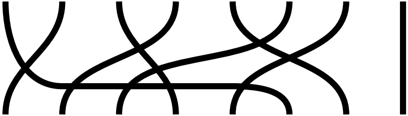
- Every graph can ... urface (proof above)
- Hence, we can talk about faces in crystals

String diagrams – the baby case

Connect eight points at the bottom with eight points at the top:

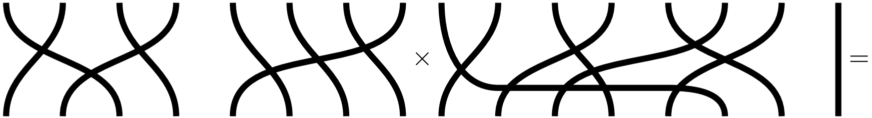
$(1243)(5876) \leftrightsquigarrow$ 

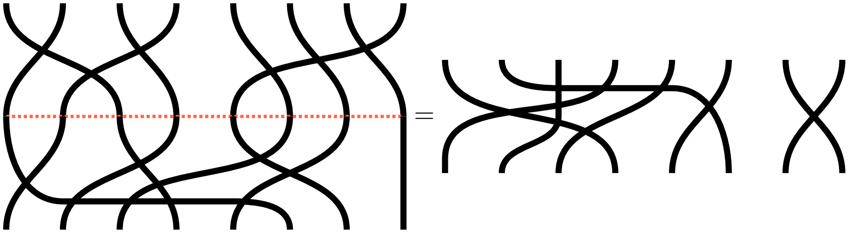
or

$(12436)(57)(8) \leftrightsquigarrow$ 

We just invented the symmetric group S_8

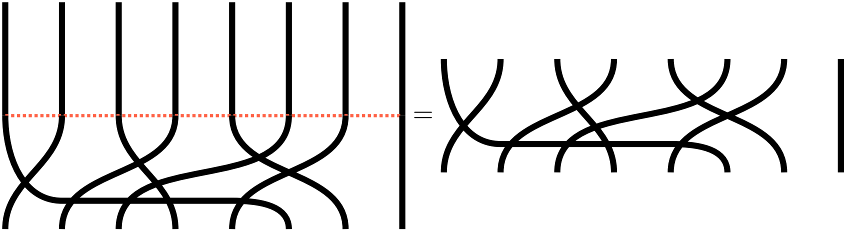
String diagrams – the baby case





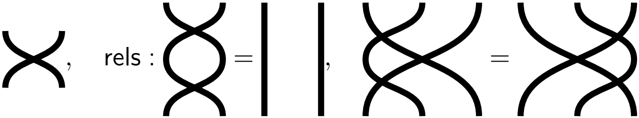
My multiplication rule for gh is "stack g on top of h"

- We clearly have $g(hf) = (gh)f$
- There is a do nothing operation $1g = g = g1$



- Generators–relations (the Reidemeister moves)

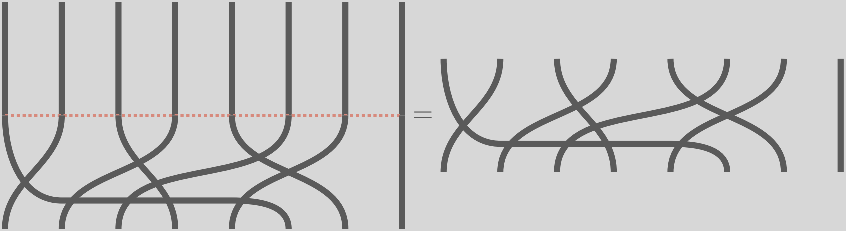
gens :



The bait

In diagram algebras relations, properties, etc. become visually clear

- We clearly have
- There is a do nothing operation $1g = g = g1$



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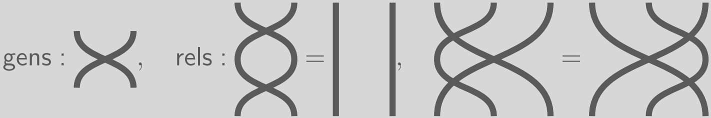

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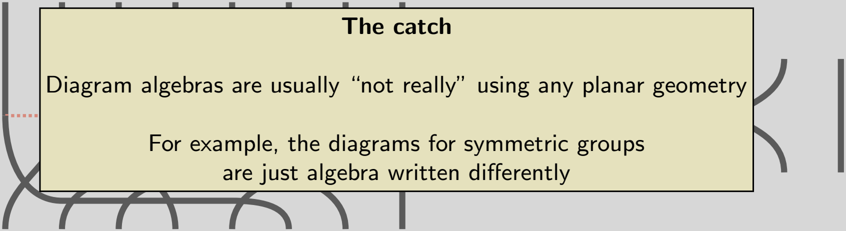
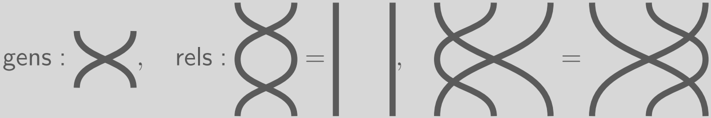


The catch

Diagram algebras are usually "not really" using any planar geometry

For example, the diagrams for symmetric groups are just algebra written differently

- Generators–relations (the Reidemeister moves)



String diagrams –

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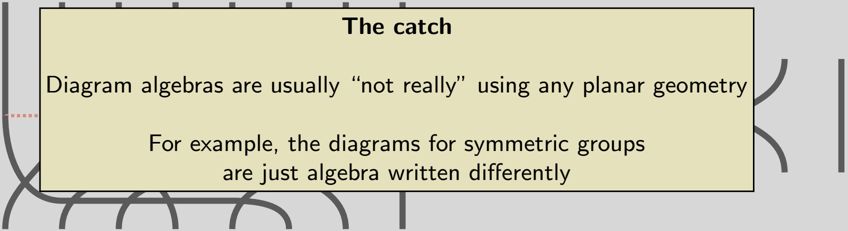

The catch

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For example, the diagrams for symmetric groups are just algebra written differently

- Gene

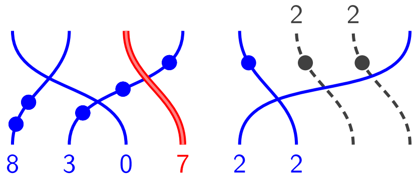
Idea (Webster ∼2012)

Define a diagram algebra that uses the distance in $\mathbb{R}^2$

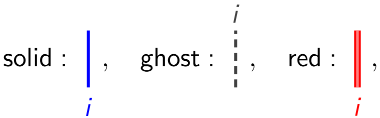
The result is called weighted KLRW=wKLRW algebra

These are "planar-geometrically symmetric group diagram algebras"

Weighted string diagrams

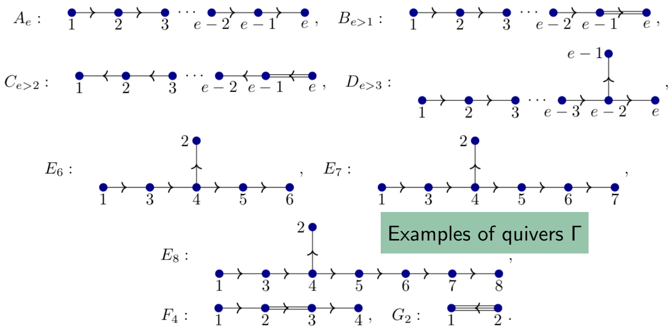


- Strings come in three types, solid, ghost and red

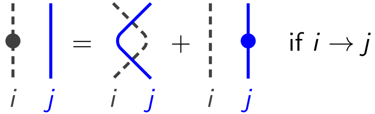


- Strings are labeled, and solid and ghost strings can carry dots
- Red strings anchor the diagram (red strings ⇜⇝ level)
- Otherwise no difference to symmetric group diagrams

Weighted string diagrams

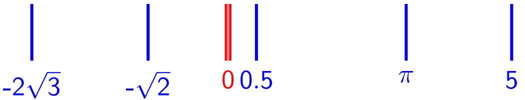


- The strings are labeled by $i \in I$ from a fixed quiver $\Gamma = (I, E)$
- The relations (that I am not going to show you ;-)) depend on $e \in E$, e.g.:



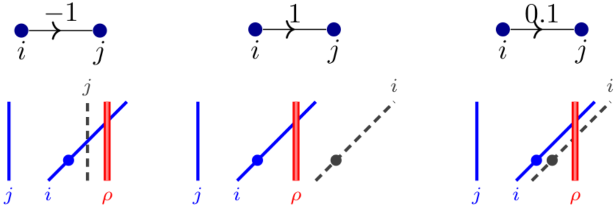
Weighted string diagrams

$X = (\text{-}2\sqrt{3}, -\sqrt{2}, 0.5, \pi, 5) \rightsquigarrow$



Weighted quiver

diagram



- Choose endpoints $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, $\rho \in \mathbb{R}^\ell$ for the solid and red strings
- Choose a weighting $\boldsymbol{\sigma}: E \to \mathbb{R}_{\neq 0}$ of the underlying graph $\Gamma = (I, E)$
- The wKLRW algebra crucially depends on these choices of endpoints! This is very different from "usual" diagram algebras

Weighted string diagrams

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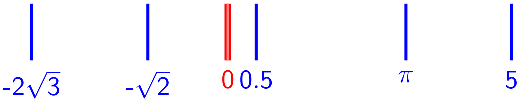

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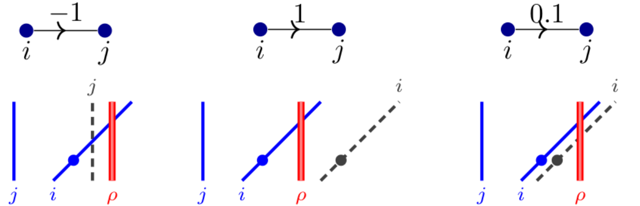


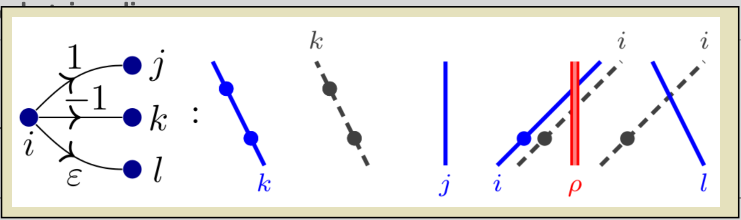
Weighting = ghost shifts

For $\epsilon\colon i \to j, \sigma_\epsilon > 0$, all solid i-strings get a ghost shifted $|\sigma_\epsilon|$ units and mimicking it
For $\epsilon\colon i \to j, \sigma_\epsilon < 0$, all solid j-strings get a ghost shifted $|\sigma_\epsilon|$ units and mimicking it

- Choose a weighting $\sigma\colon E \to \mathbb{R}_{\neq 0}$ of the underlying graph $\Gamma = (I, E)$
- The wKLR ... endpoints! This is very different from "usual" diagram algebras

This "asymmetric" definition, always shifting rightwards makes life a bit more convenient

Weight

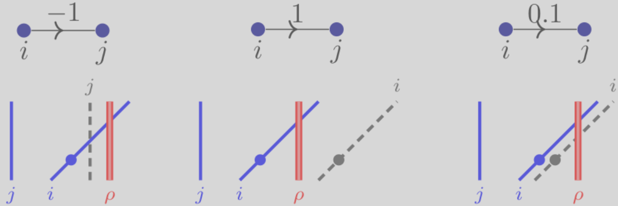


$X = ($

5

Weighted quiver

diagram



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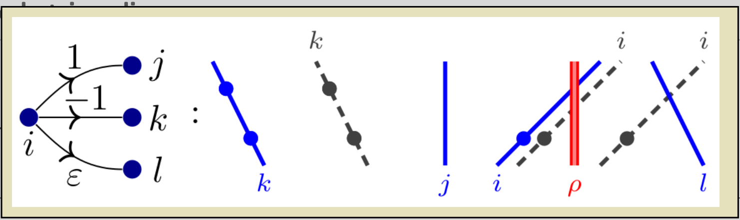
Weight…

$X = ($… 5



Weighted quiver

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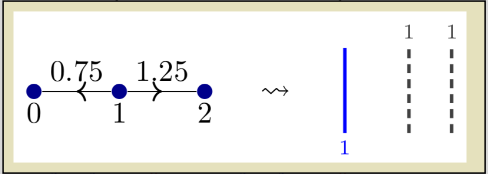

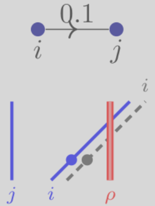

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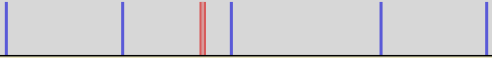
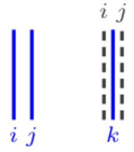
- The wKLRW algebra crucially depends on these choices of endpoints! This is very different from "usual" diagram algebras

Weighted string diagrams

$X = (-2\sqrt{3}, -\sqrt{2}, 0.5, \pi, 5) \leftrightsquigarrow$ 

The following i and j-strings are not close:



Slogan Ghosts prevent the diagrams from being scale-able as for "usual" diagram algebras

- Choose endpoints $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\rho \in \mathbb{R}^\ell$ for the solid and red strings
- Choose a weighting $\boldsymbol{\sigma}\colon E \to \mathbb{R}_{\neq 0}$ of the underlying graph $\Gamma = (I, E)$
- The wKLRW algebra crucially depends on these choices of endpoints! This is very different from "usual" diagram algebras

Weighted string diagrams

“Big” $\mathbb{C}[X]/(X-a)(X-b)$

$a{=}0, b{=}0$ $a{=}1, b{=}-1$

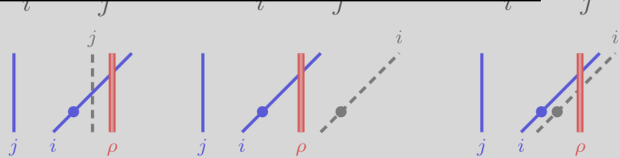
“Small” $\mathbb{C}[X]/(X^2)$ $\mathbb{C}[X]/(X^2-1)$

$a, b \leftrightsquigarrow$ Positions and ghost shifts

$X = (\text{-}2\surd$ 5

Weighted .1

diagram



- Choose endpoints $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\rho \in \mathbb{R}^\ell$ for the solid and red strings
- Choose a weighting $\boldsymbol{\sigma}\colon E \to \mathbb{R}_{\neq 0}$ of the underlying graph $\Gamma = (I, E)$
- The wKLRW algebra crucially depends on these choices of endpoints! This is very different from “usual” diagram algebras

Weighted string diagrams

"Big" $\mathbb{C}[X]/(X-a)(X-b)$

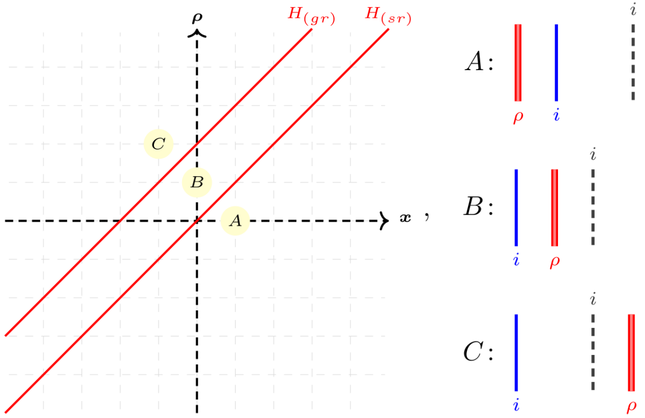
- $a=0, b=0$: "Small" $\mathbb{C}[X]/(X^2)$
- $a=1, b=-1$: $\mathbb{C}[X]/(X^2-1)$

$a, b \leftrightsquigarrow$ Positions and ghost shifts

Semisimple	Huge ghost shifts
KLR	Tiny ghost shifts
Quiver Schur	Some specific "cluster" spacing
Diagrammatic Cherednik	Ghost shifts 1
Unnamed algebras	The rest

- Choose endpoints $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\rho \in \mathbb{R}^\ell$ for the solid and red strings
- Choose a weighting $\boldsymbol{\sigma}\colon E \to \mathbb{R}_{\neq 0}$ of the underlying graph $\Gamma = (I, E)$
- The wKLRW algebra crucially depends on these choices of endpoints! This is very different from "usual" diagram algebras

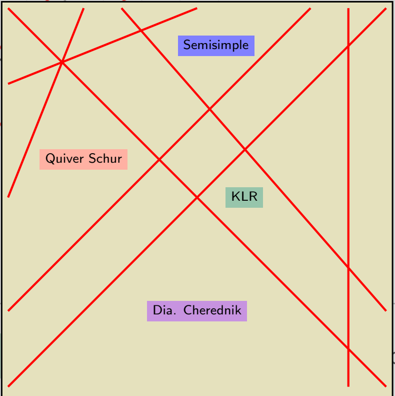
Weighted string diagrams



- There is a hyperplane arrangment (HA) associated to the wKLRW
- The hyperplanes are defined by "colliding strings" (a form of distance)

We

- Alcoves of the HA ⇒ Morita equivalence classes of wKLRW algebras
- There is a theory of translation functors
- ≈picture 1 There is an alcove for KLR, an alcove for the semisimple case etc.
- ≈picture 2 Translation functors interpolate between these algebras



This is an ≈picture

- There is a hyperpl ... o the wKLRW
- The hyperplanes ar ... form of distance)

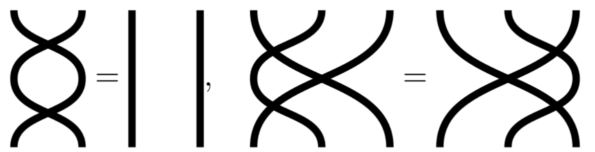
Placing strings: crystals and wKLRW

- We now play a string placing game
- Only certain “good” configurations give nice tones
- The “good” configurations come from paths in crystal graphs

Placing

wKLRW algebras are string diagram algebras associated to a quiver

with the point being the presence or absence of Reidemeister moves

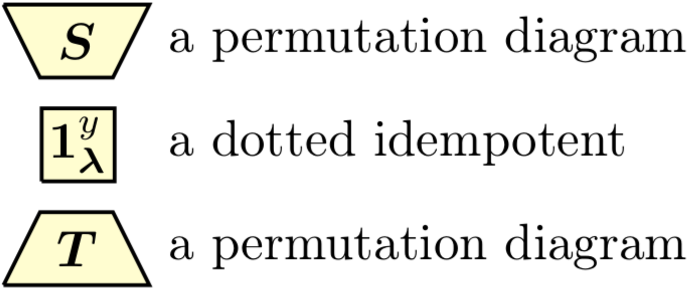


That means that sometimes strings get blocked

ng 6
ng 5
ng 4
ng 3
ng 2
ng 1

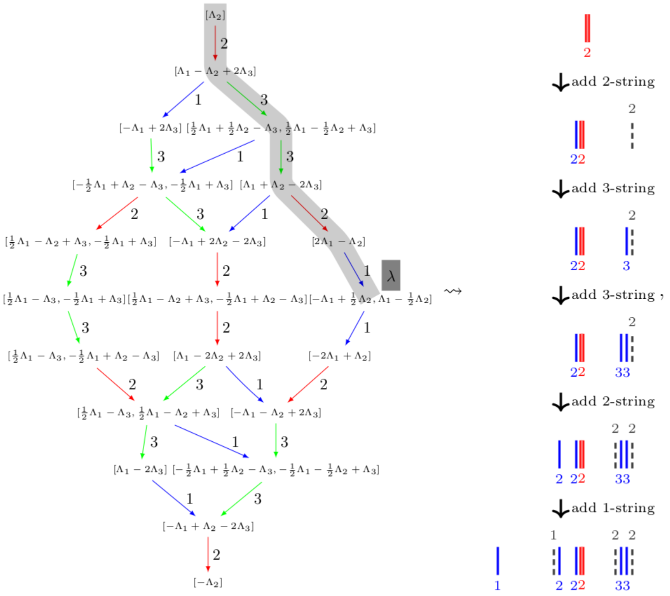
- We
- Onl
- The

Placing strings: crystals and wKLRW

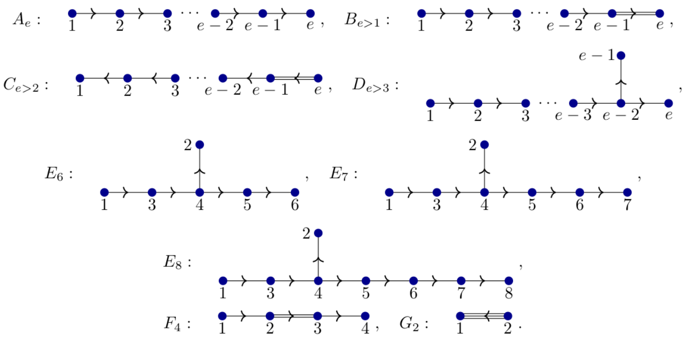


- The highest weight of the crystal tells you the starting position
- Move along a path and place strings so that they are blocked by the previous string
- This produced an idempotent 1_λ associated to a path in a crystal
- These idempotents + dots are the belts of a sandwich cellular structure; the shirt/pants are Stembridge/Sternberg permutations

Placing strings: crystals and wKLRW



Placing strings: crystals and wKLRW



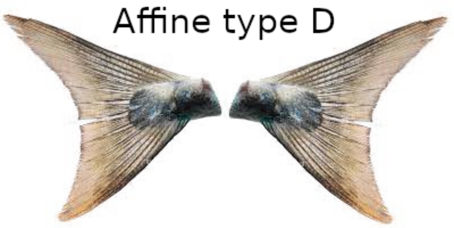
In finite types the PBW theorem for crystals implies that:

- For a fixed choice of path per vertex 1_Λ gives rise to a cell module with an associated simple
- All simples arise in this way
- Simples for different vertices are not equivalent

▶ The overall strategy to construct sandwich cellular bases
is the same for all type (but the details differ)

and for the infinite dimensional and the cyclotomic case the construction is also the same
The bases one gets are different from the ones of **Kleshchev–Loubert(–Miemietz) ∼2013**

▶ We know that the cellular bases work in several type (including finite type)
and we can use crystal combinatorics to rule out that e.g. affine types D, E work
This also uses an argument of **Ehrig, Evseev, Kleshchev–Muth ∼?**

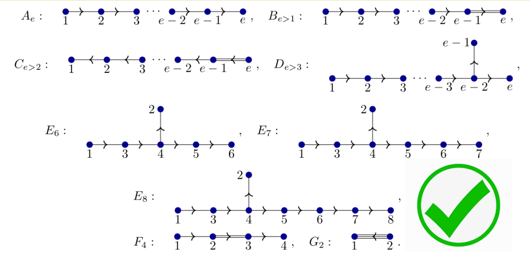


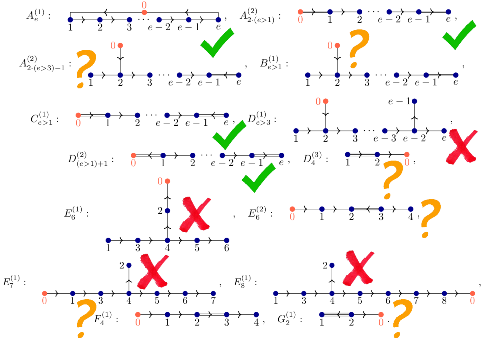
▶ The combinatorics is inspired by, but different from, constructions of
Bowman ∼2017, Ariki–Park ∼2012/2013, Ariki–Park–Speyer ∼2017

▶ Simples for different vertices are not equivalent

Placing

Overview of cellularity (strictly speaking finite type is work in progress)





In finite

- Fo … h an as
- Al
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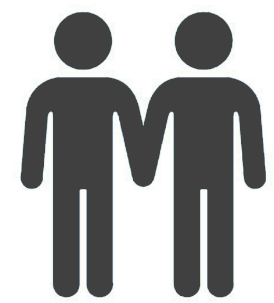
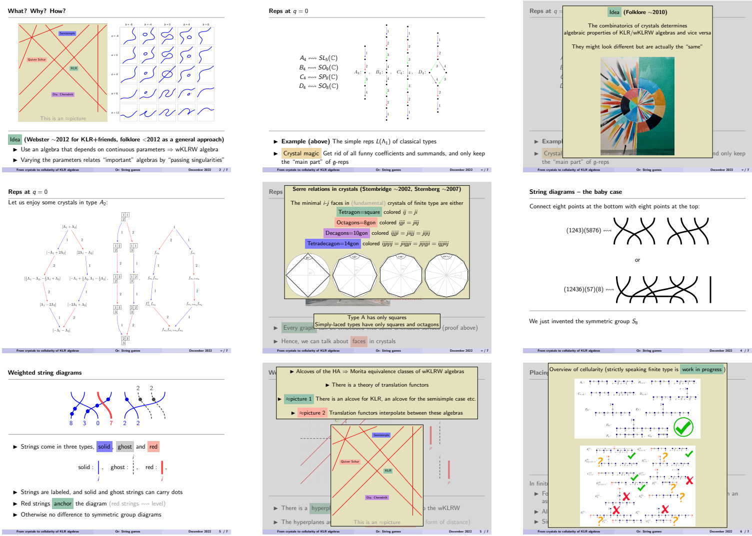
Wrap up

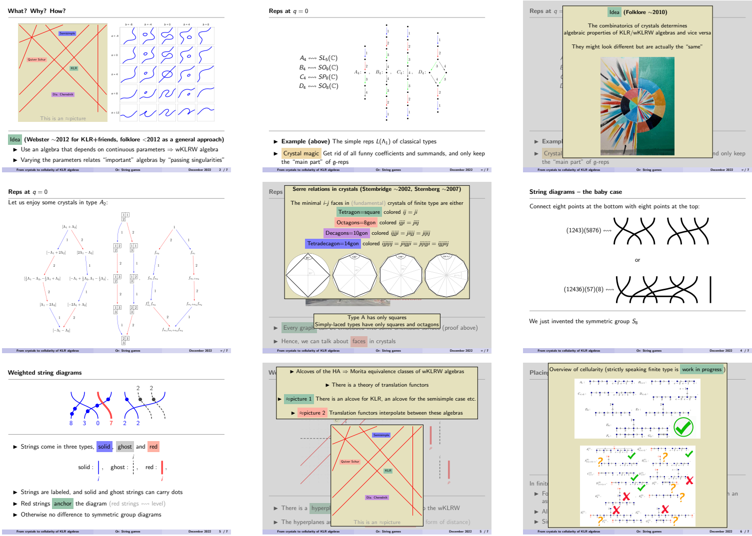
KLR + crystal



- wKLRW algebras generalize KLR algebras and friends
- wKLRW algebras interpolate between KLR algebras and friends
- Bases can be constructed from crystals (wishful thinking)
- These bases are sometimes cellular, depending on the crystal (wishful thinking)



There is still much to do...



Thanks for your attention!