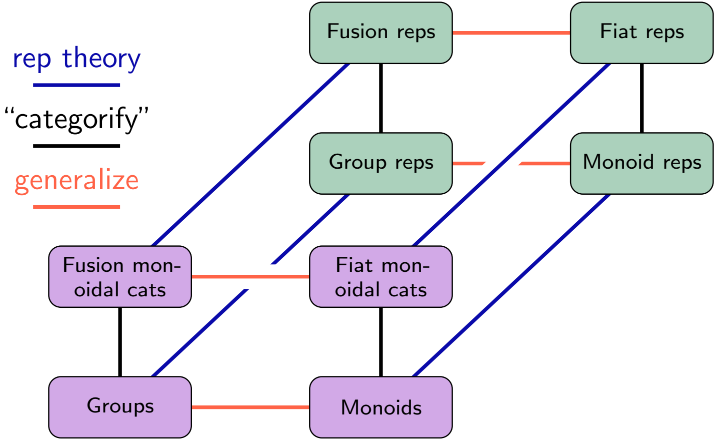
Minimal representations of monoids

Or: The smallest nontrivial

Daniel Tubbenhauer

Joint with M. Khovanov and M. Sitaraman

Where do we want to go?



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- Monoids reps have a slightly different flavor than group reps

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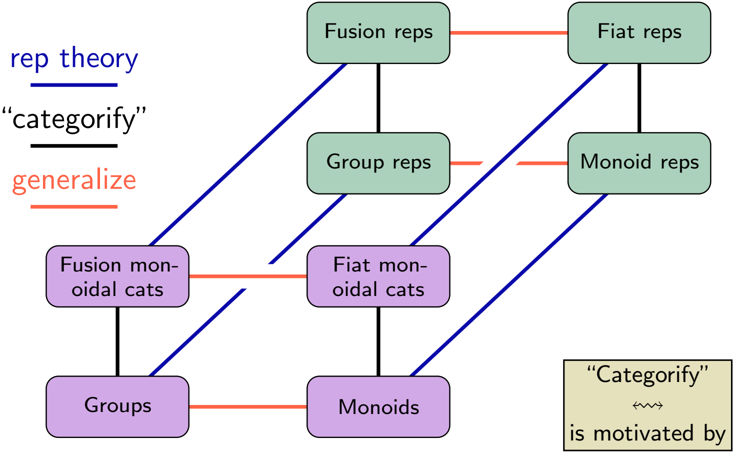

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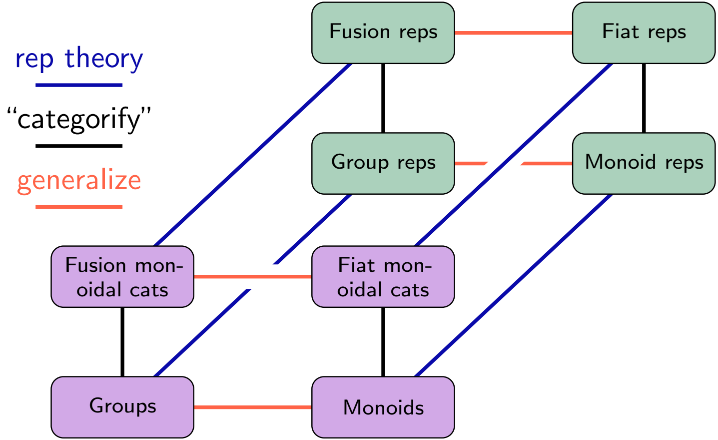

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- Goal Describe "minimal" monoid reps

Where do we want to go?

Fusion reps — Fiat reps

rep theory

You are probably asking right now: **Why monoids?**

Excellent question! Here are some biased reasons:

They are fun!

Monoids are at the heart of additive 2-representation theory (previous slide)

Monoids generalize groups but they are still better than general algebras

Ditto for their reps

Monoids are part of combinatorics, algebras are not

In cryptography it is preferable to not have a linear structure

In monoidal/tensor categories $S_n = \mathrm{End}_{\mathcal{C}}(V^{\otimes n})$ give families of monoids (more later)

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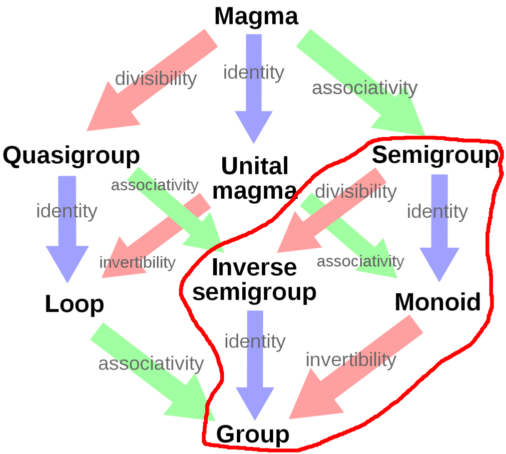
Monoids are everywhere



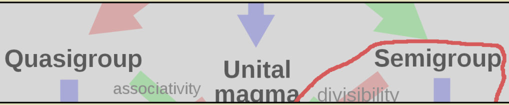
- Associativity ⇒ reasonable theory of matrix reps
- Southeast corner ⇒ reasonable theory of matrix reps

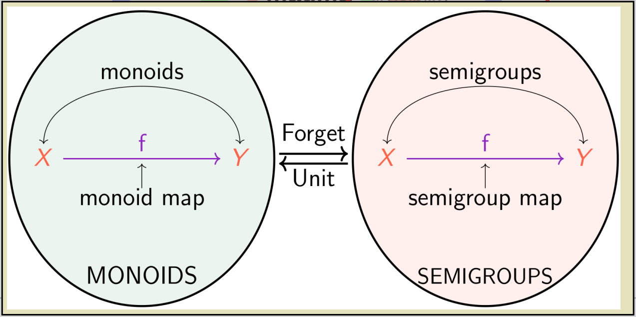
Monoi

Adjoining identities is "free" and there is no essential difference between semigroups and monoids

The main difference is monoids vs. groups

I will stick with the more familiar monoids and groups





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The point of monoid theory is to keep track of information loss

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The point of monoid theory is to keep track of information loss

Monoids appear naturally in categorification

Group-like structures

	Totality$^{\alpha}$	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Inverse semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

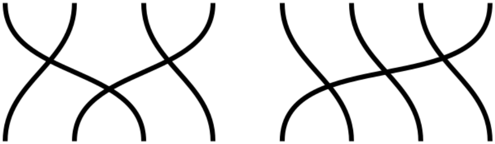
- Associativity =
- Southeast corr

Examples of monoids

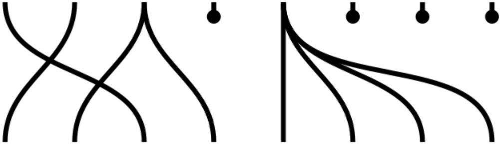
Groups

Multiplicative closed sets of matrices (these need not to be unital, but anyway)

Symmetric groups $\mathrm{Aut}(\{1, ..., n\})$

$(24138567) \leftrightsquigarrow$ 

Transformation monoids $\mathrm{End}(\{1, ..., n\})$

$(23135555) \leftrightsquigarrow$ 

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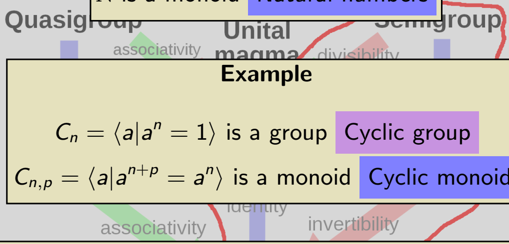


Example

$\mathbb{Z}$ is a group — Integers

$\mathbb{N}$ is a monoid — Natural numbers

Example

$C_n = \langle a | a^n = 1 \rangle$ is a group — Cyclic group

$C_{n,p} = \langle a | a^{n+p} = a^n \rangle$ is a monoid — Cyclic monoid

Example (now with notation)

$\mathfrak{S}_n = \mathrm{Aut}(\{1, ..., n\})$ is a group — Symmetric group

$\mathfrak{T}_n = \mathrm{End}(\{1, ..., n\})$ is a monoid — Transformation monoid

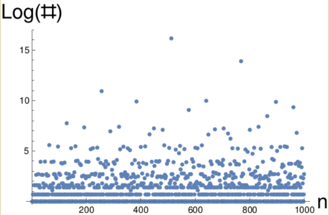
- Associativity $\Rightarrow$ reasonable theory of matrix reps
- Southeast corner $\Rightarrow$ reasonable theory of matrix reps

Mono

Finite groups are kind of random...

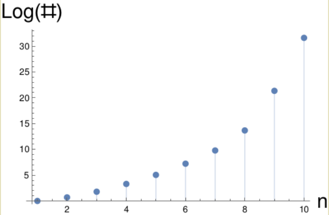
A000001 Number of groups of order n.
(Formerly M0098 N0035)

0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1,
51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1,
2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,



A058133 Number of monoids (semigroups with identity) of order n, considered to be equivalent when they are isomorphic or anti-isomorphic (by reversal of the operator).

0, 1, 2, 6, 27, 156, 1373, 17730, 858977, 1844075697, 52991253973742 (list; graph; refs; listen; history;



▶ A

▶ S

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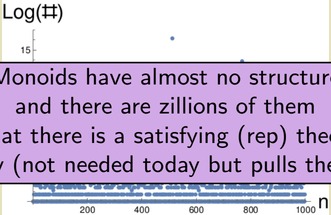

Monoids have almost no structure
and there are zillions of them
⇒ not clear that there is a satisfying (rep) theory of monoids
There is: Green's cell theory (not needed today but pulls the strings in the background)

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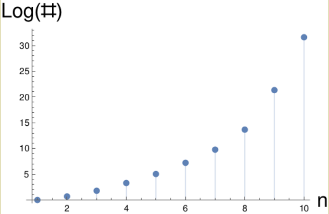
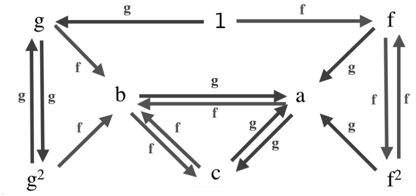


Monoids are everywhere

Cayley graphs of monoids might look weird:

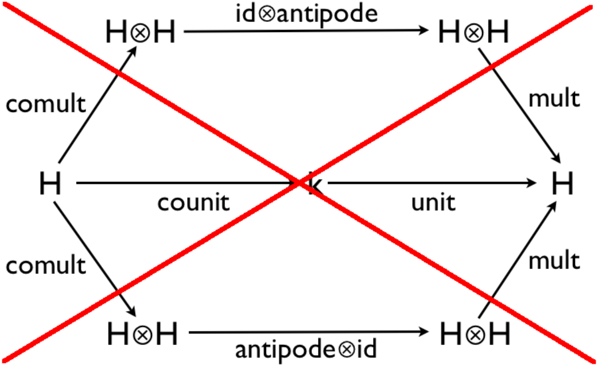


- $S_{0,...,n-1;1}$ = monoid on $\{0, ..., n-1\} \cup \{1'\}$
- $1'$ is the unit and $ab = a$ otherwise
- One can check that $\mathbb{K}S_{0,...,n-1;1}$ is a split basic algebra whose quiver Γ is of the form

$$n = 1\colon \Gamma = \bullet \quad \bullet, \quad n = 2\colon \Gamma = \bullet \to \bullet, \quad n = 3\colon \Gamma = \bullet \rightrightarrows \bullet,$$

i.e. two vertices and $n-1$ edges for $\mathbb{K}S_{0,...,n-1;1}$

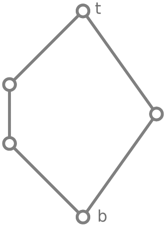
Monoids are everywhere



- In the theory of monoids the key are Green cells
- The monoid algebra $\mathbb{K}S$ is a bialgebra $\Rightarrow$ rep cat is monoidal
- Crucial $\mathbb{K}S$ is not a Hopf algebra $\Rightarrow$ rep cat is not rigid

Minimal monoids representations

or a bit more accurate:



- S = monoid, $G \subset S$ = group of units
- S has two trivial reps, called bottom and top:

$$1_b \colon S \to \mathbb{K}, \quad s \mapsto \begin{cases} 1 & \text{if } s \in G, \\ 0 & \text{else}, \end{cases} \quad 1_t \colon S \to \mathbb{K}, \quad s \mapsto 1.$$

- The name comes from the fact that simple monoid reps are partially ordered and these are at bottom/top

Minimal monoids representation

Example
S is a group
$\Leftrightarrow$
$S = G$
$\Leftrightarrow$
$1_b = 1_t$

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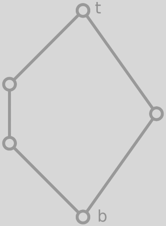

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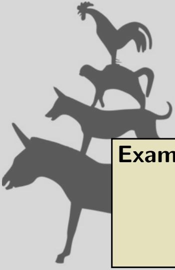

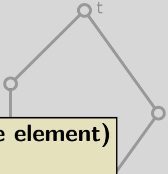

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Example (the only monoid with one element)

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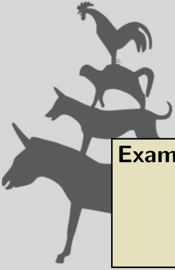

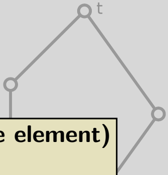

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- $S =$ monoid, G
- S has two trivia

Example (monoid 1 with two elements)

$S = S_{0;1} = \langle a | a^2 = a \rangle$ is essentially trivial

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$1_b \neq 1_t$ are the only simple S-reps

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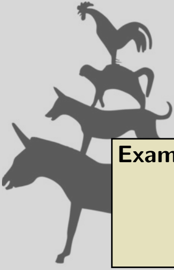

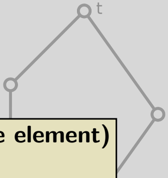

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Example (monoid 2 with two elements)

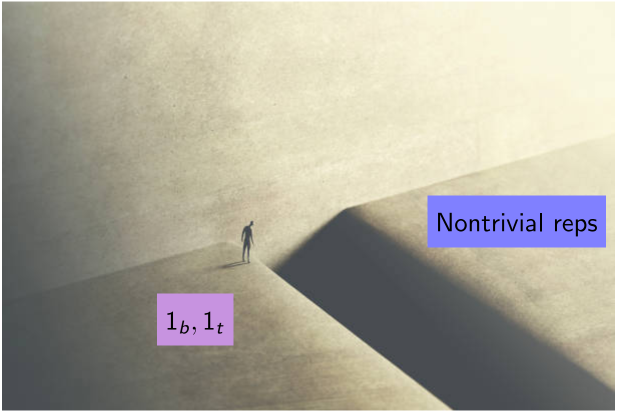
$S = \langle a | a^2 = 1 \rangle$ (this is $\mathbb{Z}/2\mathbb{Z}$)

$\Rightarrow$

$1_b = 1_t$ and $a \mapsto -1$ are the only simple S-reps

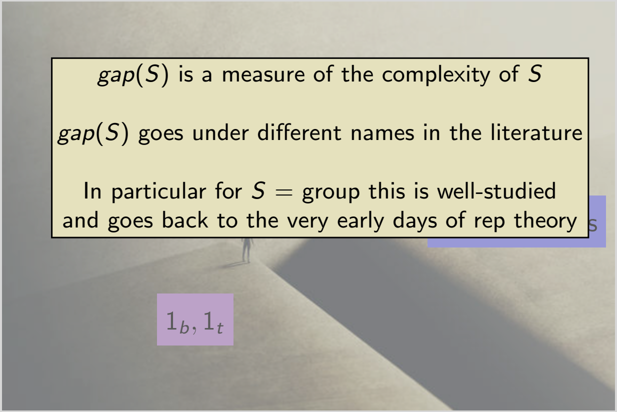
- The name con… e partially ordered and these are …

Minimal monoids representations



- Call all S-reps $1_t^{\oplus m} \oplus 1_b^{\oplus n}$ trivial
- Rep gap $gap_{\mathbb{K}}(S)$ = smallest dim of a nontrivial S-rep over $\mathbb{K}$; $gap_* = \min$ of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

Minimal monoids representations



gap(S) is a measure of the complexity of S

gap(S) goes under different names in the literature

In particular for $S =$ group this is well-studied and goes back to the very early days of rep theory

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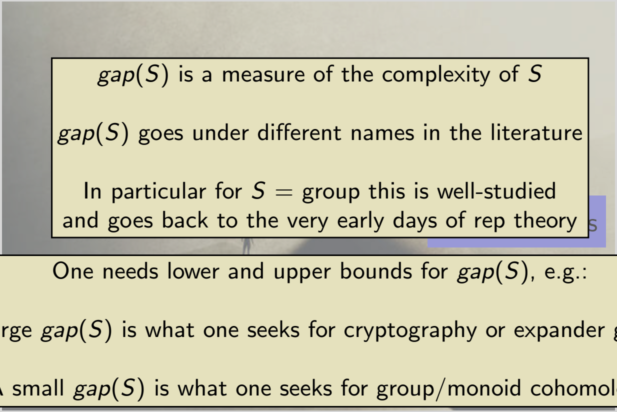

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One needs lower and upper bounds for *gap*(S), e.g.:

A large *gap*(S) is what one seeks for cryptography or expander graphs

A small *gap*(S) is what one seeks for group/monoid cohomology

- Call all S-reps $1_t^{\oplus m} \oplus 1_b^{\oplus n}$ trivial
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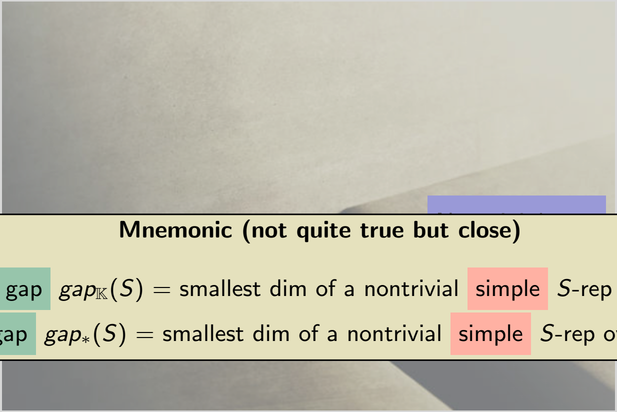

Mnemonic (not quite true but close)

Rep gap $gap_{\mathbb{K}}(S)$ = smallest dim of a nontrivial simple S-rep over $\mathbb{K}$

Rep gap $gap_{*}(S)$ = smallest dim of a nontrivial simple S-rep over all $\mathbb{K}$

- Call all S-reps $1_t^{\oplus m} \oplus 1_b^{\oplus n}$ trivial
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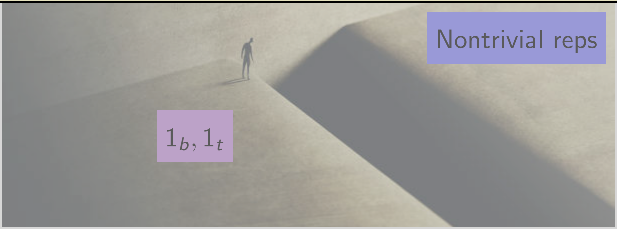
Minimal monoids representations

Example/convention

For $S = \{1\}$ we define $gap(S) = 0$

For $S = S_{0;1} = \{0, 1\}$ we define $gap(S) = 0$

Why? These are the only monoids without any nontrivial reps so $gap(S)$ would be infinite, but that is silly...



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Example (groups)

For $S = \mathbb{Z}/2\mathbb{Z}$ we have $gap_{\mathbb{C}}(S) = 1$

For $S = \mathfrak{S}_n = Aut(\{1, ..., n\})$ we have $gap_{\mathbb{C}}(S) = 1$

For $S =$ Monster we have $gap_{\mathbb{C}}(S) = 196883$ **(Griess ∼1980 and others)**

For $S = SL_2(\mathbb{F}_p)$ we have $gap_{\mathbb{C}}(S) \geq \frac{p-1}{2}$ **(Frobenius ∼1900)**

For $S = \mathbb{Z}/2\mathbb{Z}$ we have $gap_*(S) = 1$

For $S = \mathfrak{S}_n = Aut(\{1, ..., n\})$ we have $gap_*(S) = 1$

For $S =$ Monster we have $gap_*(S) \leq 196882$ **(Griess–Smith ∼1994)**

For $S = SL_2(\mathbb{F}_p)$ we have $gap_*(S) = 2$ since we can act on $\mathbb{F}_p^2$

- Ca
- R $=$ min of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

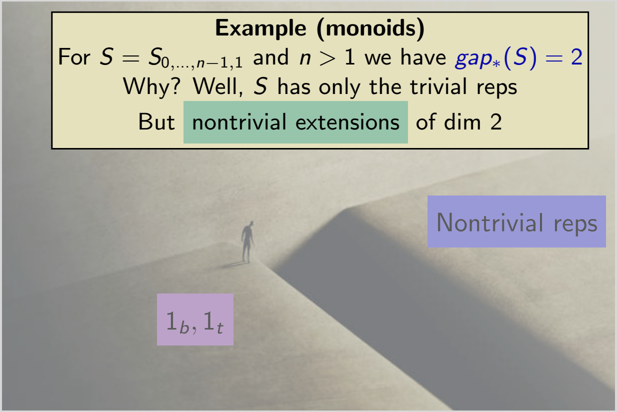
Minimal monoids representations

Example (monoids)

For $S = S_{0,\dots,n-1,1}$ and $n > 1$ we have $gap_*(S) = 2$

Why? Well, S has only the trivial reps

But nontrivial extensions of dim 2



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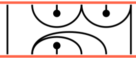
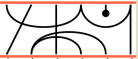
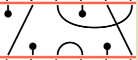
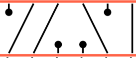
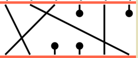
Example (monoids)

There will be some results for diagram monoids momentarily

Symbol	Diagrams	Symbol	Diagrams
pPa_n		Pa_n	
Mo_n		RoBr_n	
TL_n		Br_n	
pRo_n		Ro_n	
pS_n		S_n	

- Call all

- Rep gap $gap_{\mathbb{K}}(S)$ = smallest dim of a nontrivial S-rep over $\mathbb{K}$; gap_* = min of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

M

Honorable mentions

Old Faithful

Eruption of Old Faithful in 1948

Alternatively, and studied in group theory since the early days (under various names) and by e.g. **Mazorchuk–Steinberg ∼2011** in monoid theory one could use faithfulness as a measure of complexity (using the same notation):

Faithfulness $faith(S)$ = smallest dim of a faithful rep

- Call all S-reps $1_t^{\oplus m} \oplus 1_b^{\oplus n}$ trivial
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Examples

For $S = \mathfrak{S}_n = \mathrm{Aut}(\{1,...,n\})$ for $n \geq 5$ we have $\mathit{faith}_{\mathbb{C}}(S) = n-1$ **(Burnside ∼1902)**
For $S = \mathfrak{T}_n = \mathrm{End}(\{1,...,n\})$ we have $\mathit{faith}_{\mathbb{C}}(S) = n$ **(Mazorchuk–Steinberg ∼2011)**

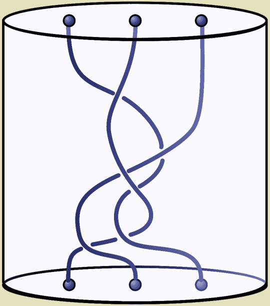
For $S = \mathfrak{S}_n = \mathrm{Aut}(\{1,...,n\})$ for $n \geq 5$ we have $\mathit{faith}_{*}(S) = n-2$ **(Dickson ∼1908)**
For $S = \mathfrak{T}_n = \mathrm{End}(\{1,...,n\})$ we have $\mathit{faith}_{*}(S) = ??$

Theorem (easy)

Under some silly nontriviality assumptions on S:

$$gap(S) \leq faith(S) \leq |S|$$

Example (infinite group but still...)



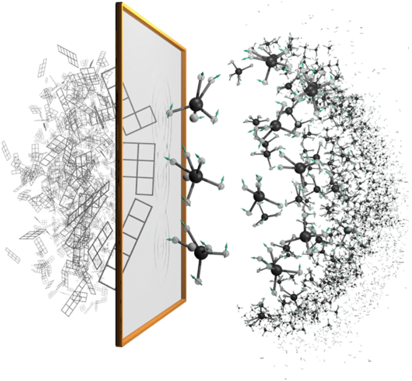
For the braid group Br_n on n strands we have $gap_{\mathbb{Q}(q,t)}(Br_n) \leq n-1$, $faith_{\mathbb{Q}(q,t)}(Br_n) \leq n(n-1)/2$

dim Burau $= n-1$, dim LKB $= n(n-1)/2$

- Call all S-re
- Rep gap g r $\mathbb{K}$; $gap_* =$ min of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

Rep gap and monoidal categories

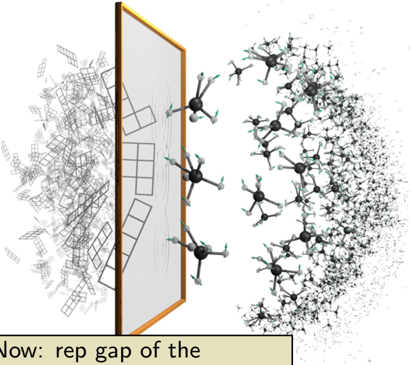
Schur–Weyl duality relates two objects



- For any monoidal category $\mathcal{C}$ we get a family of monoids $S_n = \mathrm{End}_{\mathcal{C}}(V^{\otimes n})$
- Schur–Weyl duality suggests that S_n should have a big rep gap
- Dim simple of S_n "$\Leftrightarrow$" # of indecomposables in $V^{\otimes n}$ and these grow fast

Rep gap and monoidal categories

Schur–Weyl duality relates two objects



Now: rep gap of the
"easiest Schur–Weyl duality monoid"

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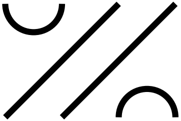
Rep gap and monoidal categories

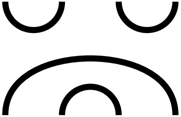
Connect 4 points at the bottom with 4 points at the top without crossings, potentially turning back:

$$\big\{\{1,-3\},\{2,-4\},\{3,4\},\{-1,-2\}\big\} \leftrightsquigarrow$$



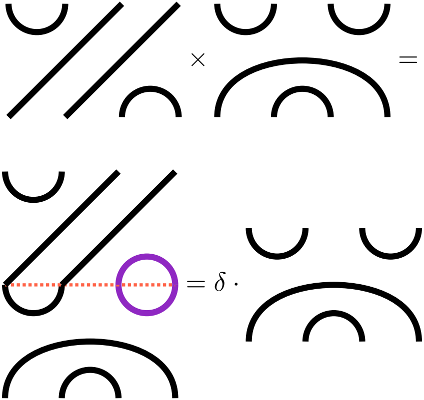
or

$$\big\{\{1,4\},\{2,3\},\{-1,-2\},\{-3,-4\}\big\} \leftrightsquigarrow$$



This is the Temperley–Lieb (TL) monoid TL_4 on $\{1,...,4\} \cup \{-1,...,-4\}$
In combinatorics, these are crossingless perfect matchings

Rep gap and monoidal categories

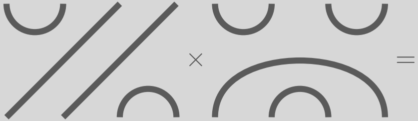


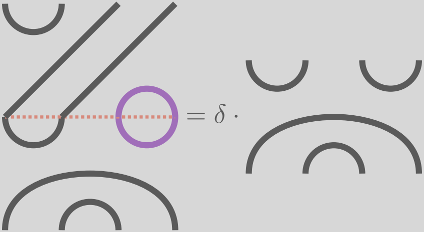
Fix some field $\mathbb{K}$ and $\delta \in \mathbb{K}$, evaluate circles to $\delta \Rightarrow$ TL algebra $TL_4(\delta)$

The TL monoid is the non-linear version of $TL_4(1)$

Rep gap and m

The TL monoid TL_n arise (kicking out scalars) under Schur–Weyl duality as
$TL_n \cong \mathrm{End}_{U_q(gl_2)}\big((\mathbb{C}_q^2)^{\otimes n}\big)$ for $-q-q^{-1}=1$





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The TL algebra goes back to **Rumer–Teller–Weyl** $\sim$**1932**

Zur Theorie der Spinvalenz.

Von

Georg Rumer in Moskau.

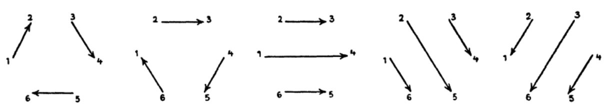
Vorgelegt von H. Weyl in der Sitzung am 22. Juli 1932.

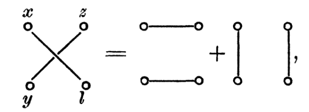
Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten.

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

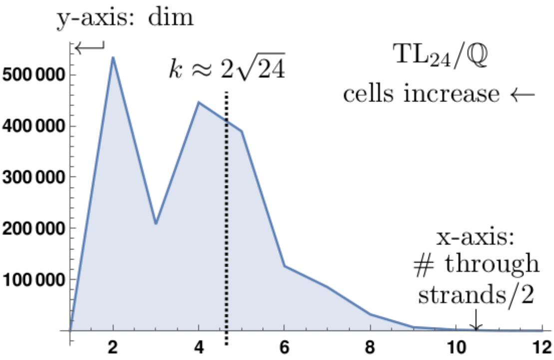
Vorgelegt von H. Weyl in der Sitzung am 28 Oktober 1932.





It has been rediscovered many times

Rep gap and monoidal categories



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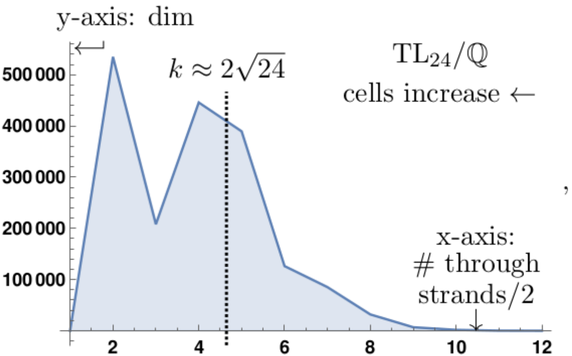
- **Fact** There is one simple TL_n-rep for each through strand $i \in \{n, n-2, ...,\}$
- **Fact** The simple dims are known recursively, see e.g. **Andersen ∼2017, Spencer ∼2020**
- **Fact** The simple dims behave as above, see e.g. **A computer ∼2021**

Rep gap and monoidal categories



On can define a truncation TL_n^k to get rid of the small reps

- Fact There is one simple TL_n-rep for each through strand $i \in \{n, n-2, ...,\}$
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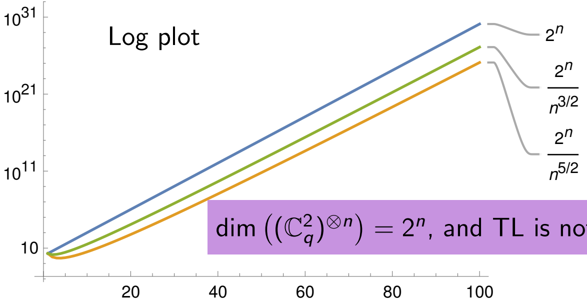
Rep gap and monoidal categories



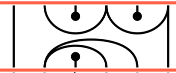
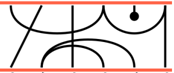
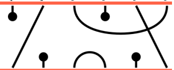
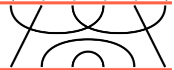
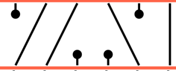
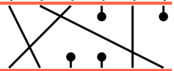
$\dim\left((\mathbb{C}_q^2)^{\otimes n}\right) = 2^n$, and TL is not far away!

Theorem For $0 \leq k \leq 2\sqrt{n}$ we have

$$rep_{\mathbb{Q}}(TL_n^k) \geq \frac{4}{(n+2\sqrt{n}+2)(n+2\sqrt{n}+4)}\binom{n}{\frac{n}{2}-\sqrt{n}} \in \Theta\left(2^n n^{-5/2}\right)$$

$$faith_{\mathbb{Q}}(TL_n^k) \geq \frac{6}{n+4}\binom{n}{\lfloor\frac{n}{2}-1\rfloor} \in \Theta\left(2^n n^{-3/2}\right)$$

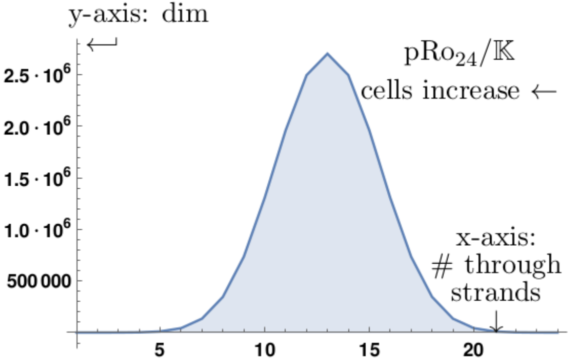
Rep gap and monoidal categories

Symbol	Diagrams	Symbol	Diagrams
pPa_n		Pa_n	
Mo_n		RoBr_n	
TL_n		Br_n	
pRo_n		Ro_n	
pS_n		S_n	

Summary

- Similar formulas hold for *gap* and *faith* but details are unknown
- The rep gap of monoids from monoidal categories is often large
- This is in particularly true for most of the "Schur–Weyl monoids" above

Rep gap and monoidal categories



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Summary

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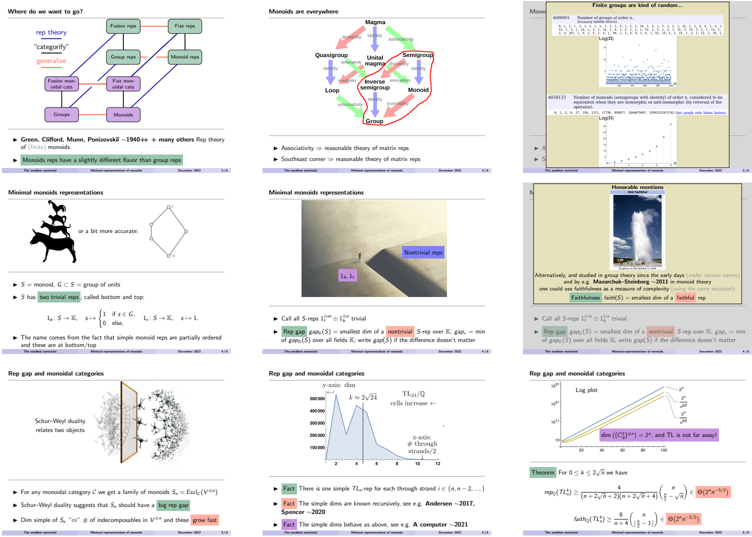
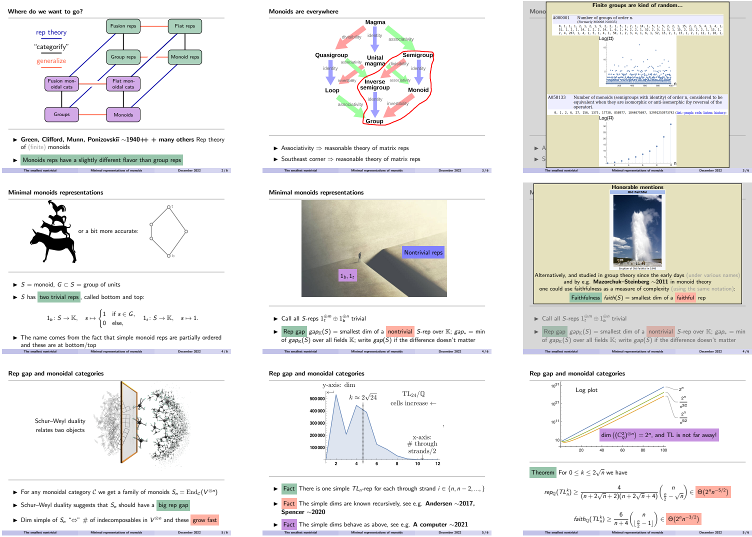


There is still much to do...



Thanks for your attention!