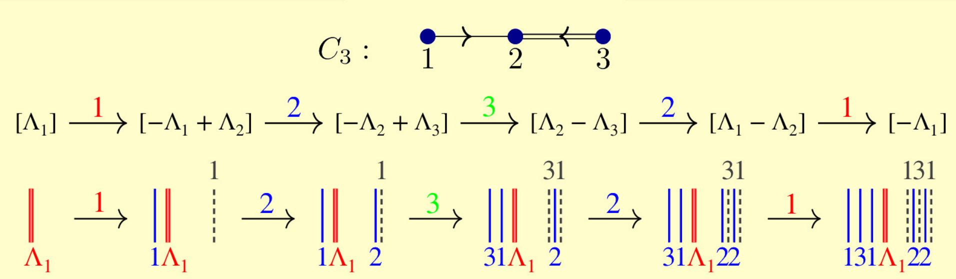
From crystals to cellularity of wKLRW algebras

Or: From path to strings

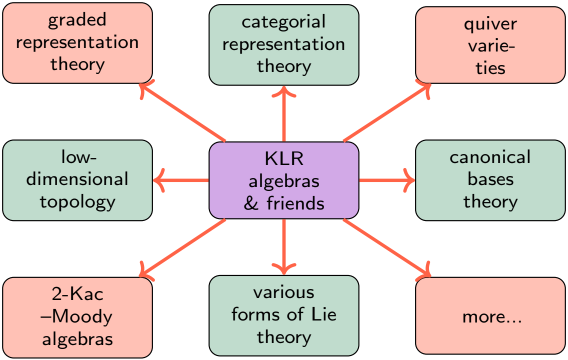
Daniel Tubbenhauer



Joint with Andrew Mathas

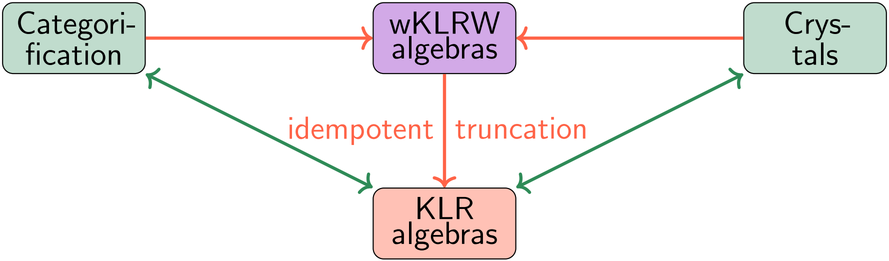
December 2022

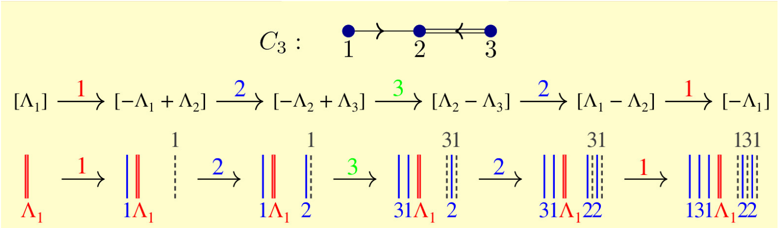
What? Why? How?



- **Khovanov–Lauda–Rouquier ∼2008 + many others** (including many people here) KLR algebras are at the heart of categorical representation theory
- Problem These are actually really complicated!
- Goal Try to find nice ("cellular") bases for them

What? Why? How?





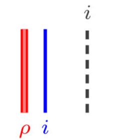
Use a richer combinatorics which is somewhat easier although more sophisticated

What? Cat fic s- s

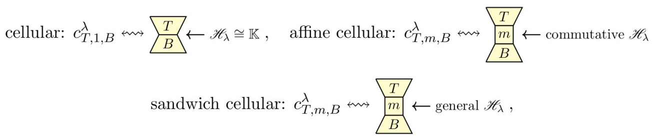
In a **60 minute talk** I would tell you about

beautiful diagrammatics, powerful abstract nonsense and partnership:

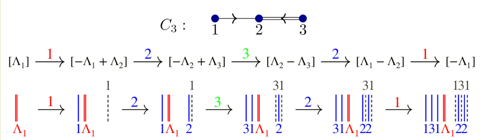
1) The diagram combinatorics beautiful



2) Sandwich cellularity powerful



3) Idempotents and crystals partners



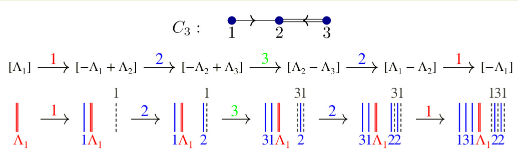
Use a icated

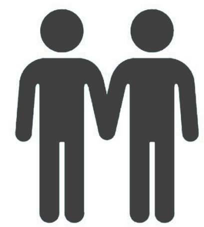
Categori-fication

Crys-tals

In a **30 minute talk** I will tell you about partnership:

3) Idempotents and crystals partners





$[\Lambda_1] \xrightarrow{1}$ … $\xrightarrow{1} [-\Lambda_1]$

Use a richer combinatorics which is somewhat easier although more sophisticated

Reps at $q = 0$

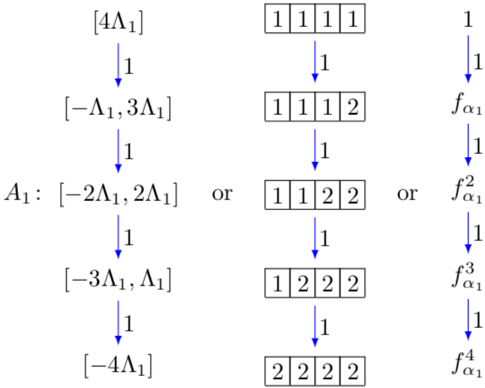
$$
A_4: \quad
\begin{array}{c}
[\Lambda_1] \\
\big\downarrow 1 \\
[-\Lambda_1 + \Lambda_2] \\
\big\downarrow 2 \\
[-\Lambda_2 + \Lambda_3] \\
\big\downarrow 3 \\
[-\Lambda_3 + \Lambda_4] \\
\big\downarrow 4 \\
[-\Lambda_4]
\end{array}
$$

- In this talk, $\mathfrak{g}$ is some Kac–Moody algebra with Chevalley generators e_i, f_i
- In essence, a crystal is a direct graph with colored edges, and it is the combinatorial shadow of a $\mathfrak{g}$-rep

vertices $\leftrightsquigarrow$ weight spaces

colored edges $\leftrightsquigarrow$ action of the f_i

Reps at $q = 0$



- **Example (above)** The simple $\mathfrak{sl}_2$-rep $Sym^4\mathbb{C}^2$ via the vanilla, tableaux, PBW flavor
- Crystal magic Get rid of all funny coefficients and summands, and only keep the "main part" of $\mathfrak{g}$-reps

Reps at $q = 0$

$$
A_4\colon
\begin{array}{c}
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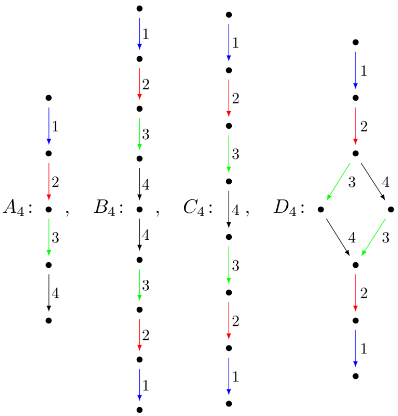
[-\Lambda_4]
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\boxed{1} \\
\downarrow 1 \\
\boxed{2} \\
\downarrow 2 \\
\boxed{3} \\
\downarrow 3 \\
\boxed{4} \\
\downarrow 4 \\
\boxed{5}
\end{array}
\quad \text{or} \quad
\begin{array}{c}
1 \\
\downarrow 1 \\
f_{\alpha_1} \\
\downarrow 2 \\
f_{\alpha_1+\alpha_2} \\
\downarrow 3 \\
f_{\alpha_1+\alpha_2+\alpha_3} \\
\downarrow 4 \\
f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}
\end{array}
$$

- **Example (above)** The simple $\mathfrak{sl}_5$-rep $\mathbb{C}^5$ via the vanilla, tableaux, PBW flavor
- Crystal magic Get rid of all funny coefficients and summands, and only keep the “main part” of $\mathfrak{g}$-reps

Reps at $q = 0$

$$A_4 \leftrightsquigarrow SL_5(\mathbb{C})$$
$$B_4 \leftrightsquigarrow SO_9(\mathbb{C})$$
$$C_4 \leftrightsquigarrow SP_8(\mathbb{C})$$
$$D_4 \leftrightsquigarrow SO_8(\mathbb{C})$$



- **Example (above)** The simple reps $L(\Lambda_1)$ of classical types
- Crystal magic Get rid of all funny coefficients and summands, and only keep the “main part” of $\mathfrak{g}$-reps

Reps at $q =$

Crystals come in many flavors:

Vanilla Works in general

Tableaux Works for classical types ABCD

PBW Works for finite types

The point Any flavor has different combinatorics

A_4

B_4

C_4

D_4

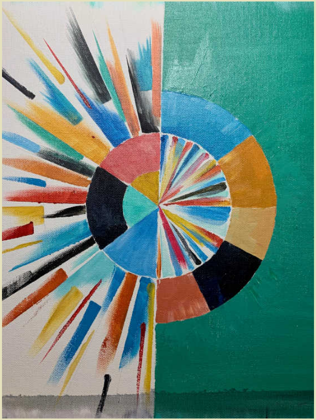
- **Example**
- Crystal magic Get rid of all funny coefficients and summands, and only keep the "main part" of $\mathfrak{g}$-reps

Reps at $q =$

Idea

The combinatorics of crystals determines
algebraic properties of KLR/wKLRW algebras and vice versa

They might look different but are actually the "same"



A

B

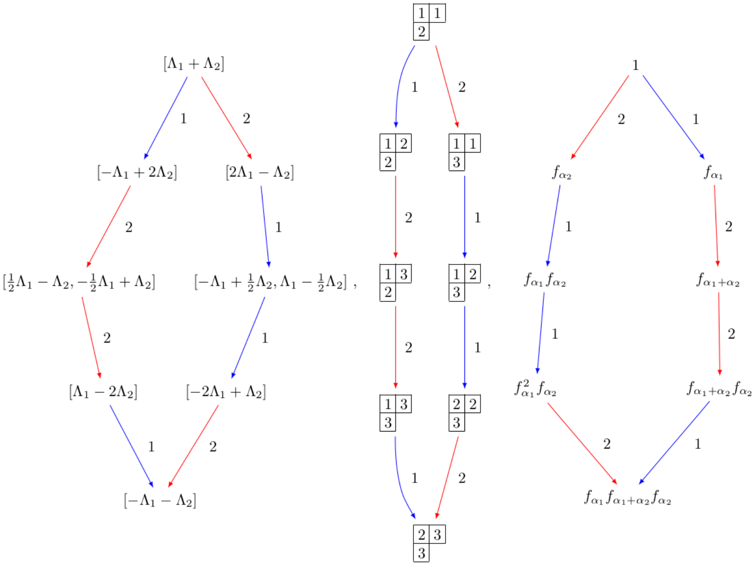
C

D

- **Exampl**
- Crystal ... nd only keep the "main part" of $\mathfrak{g}$-reps

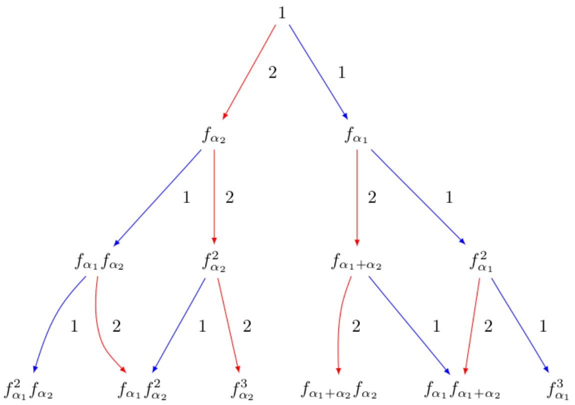
Reps at $q = 0$

Let us enjoy some crystals in type A_2:



Reps at $q = 0$

$q = 0$
of the
PBW theorem:

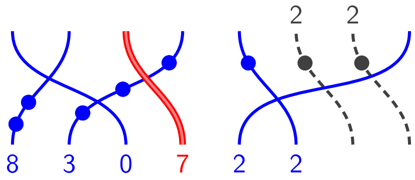


- In finite type one can cut out all crystals from a general PBW crystal
- Idea If the partnership between crystals and KLR algebras works, then finite type KLR algebra should be quite special

Placing strings: crystals and KLR (of level one – for convenience only)

- We now play a string placing game
- Only certain "good" configurations give nice tones
- The "good" configurations come from paths in crystal graphs

Placing strings: crystals and KLR (of level one – for convenience only)



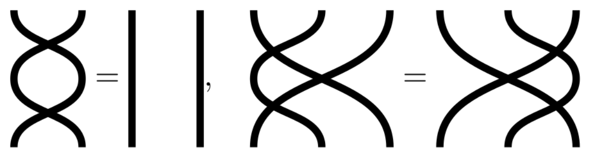
- Strings come in three types, solid, ghost and red

- Strings are labeled by simple roots, and solid and ghost strings can carry dots
- Red strings anchor the diagram (# red strings ⇝ level)
- Otherwise no difference to symmetric group diagrams
- Get wKLRW diagrams: "solid string $= f_i$, red strings $=$ highest weight"

Placing ... nly)

wKLRW algebras are string diagram algebras associated to a quiver

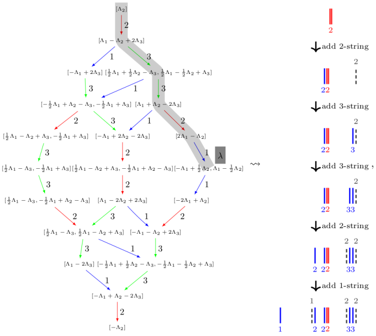
with the point being the presence or absence of Reidemeister moves



That means that sometimes strings get blocked

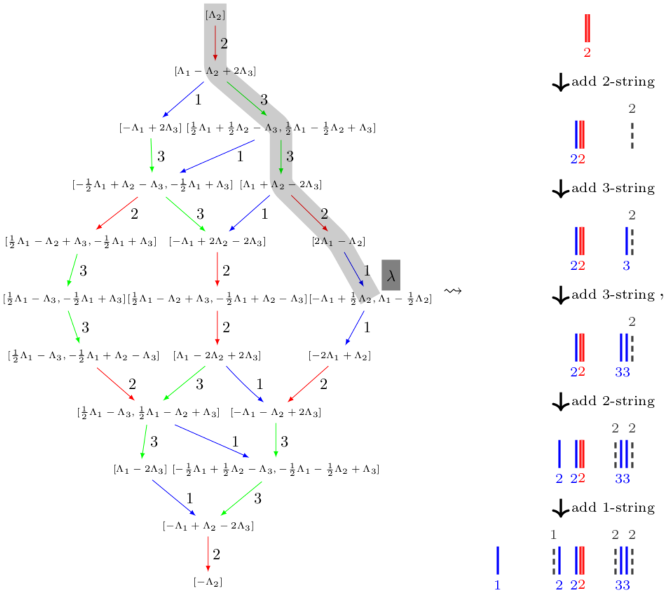
- Stri
- Stri ... arry dots
- Red
- Oth
- Get ... ght"

Placing strings: crystals and KLR (of level one – for convenience only)

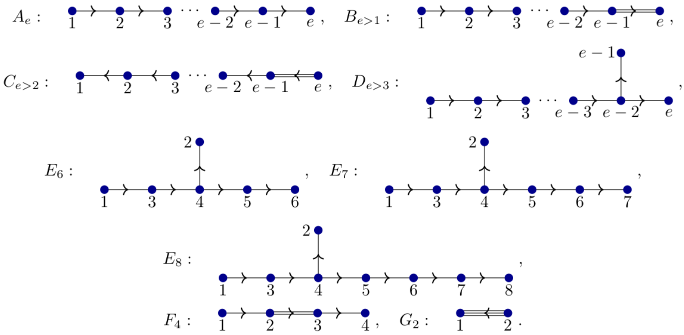


- The highest weight of the crystal tells you the starting position i.e. the red string placement
- Fix a path and move along it, while doing so place strings so that they are blocked by the previous string
- This produced an idempotent 1_Λ associated to a crystal Λ

Placing strings: crystals and KLR (of level one – for convenience only)



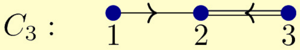
Placing strings: crystals and KLR (of level one – for convenience only)



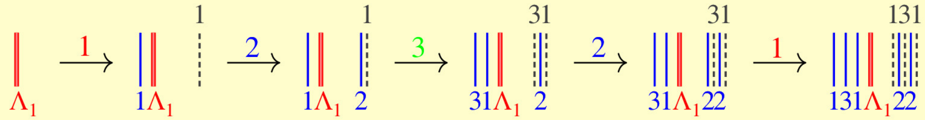
In finite types the PBW theorem for crystals implies that:

- For a fixed choice of path per vertex 1_Λ gives rise to a cell module with an associated simple
- All simples arise in this way
- Simples for different vertices are not equivalent

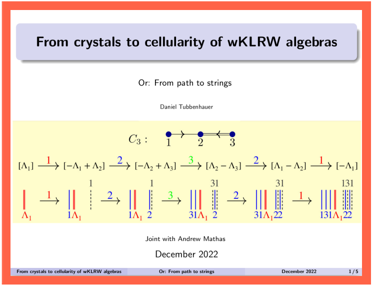
Placing strings: crystals and KLR (of level one – for convenience only)



$$[\Lambda_1] \xrightarrow{1} [-\Lambda_1+\Lambda_2] \xrightarrow{2} [-\Lambda_2+\Lambda_3] \xrightarrow{3} [\Lambda_2-\Lambda_3] \xrightarrow{2} [\Lambda_1-\Lambda_2] \xrightarrow{1} [-\Lambda_1]$$



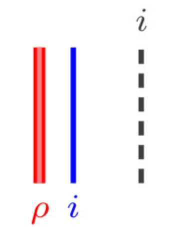
This was the thumbnail

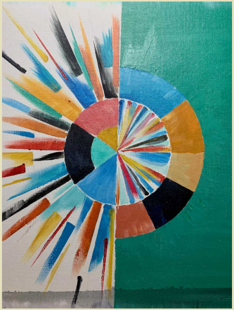


This, in finite type we get an explicit bijection between

vertices of the crystal graph and simples/projectives of the wKLRW algebras

$\leftrightsquigarrow$



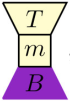


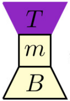
Plac

$[\Lambda_1]$ Λ_1 $[-\Lambda_1]$ 131 22

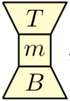
A beefed-up version (stated for completeness) is:

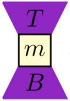
Paths in crystals give sandwich cellular bases

$\mathbb{K}\mathcal{L}_\lambda \cong \mathbb{K}\mathcal{T}(\lambda) \otimes_{\mathbb{K}} \mathscr{H}_\lambda \leftrightsquigarrow$ 

, $\mathbb{K}\mathcal{R}_\lambda \cong \mathscr{H}_\lambda \otimes_{\mathbb{K}} \mathbb{K}\mathcal{B}(\lambda) \leftrightsquigarrow$ 

,

$\mathcal{J}_\lambda \cong \mathbb{K}\mathcal{T}(\lambda) \otimes_{\mathbb{K}} \mathscr{H}_\lambda \otimes_{\mathbb{K}} \mathbb{K}\mathcal{B}(\lambda) \leftrightsquigarrow$ 

, $\mathbb{K}\mathcal{H}_\lambda \cong \mathscr{H}_\lambda \leftrightsquigarrow$ 

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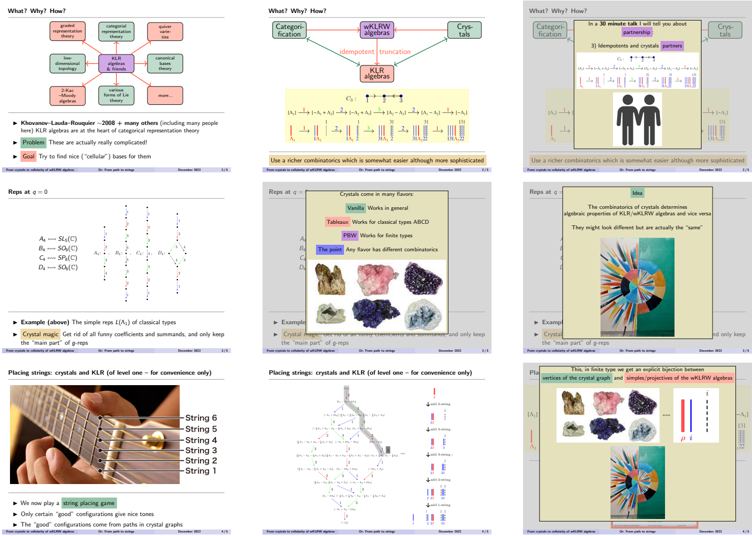
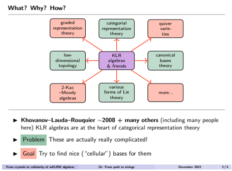
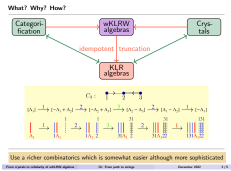
This works in all finite types but also beyond

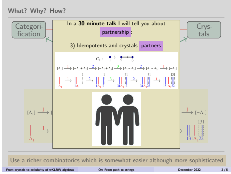
Conversely if the crystals do not satisfy certain properties then cellularity fails

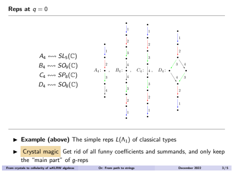


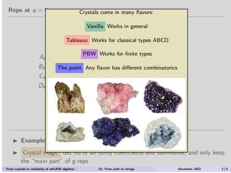
There is still much to do...

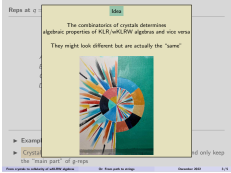


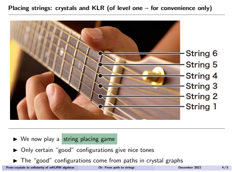


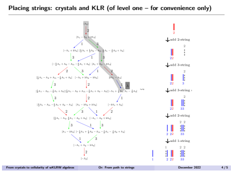


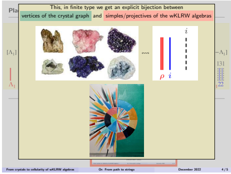












Thanks for your attention!