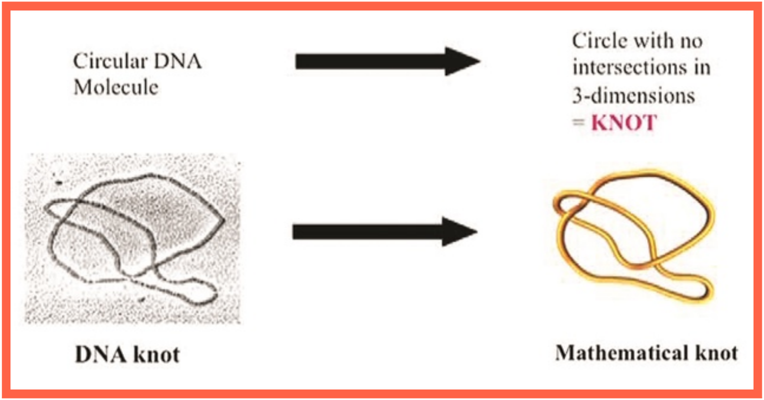
Knots and algebra

Or: Quantum algebra = geometry + algebra

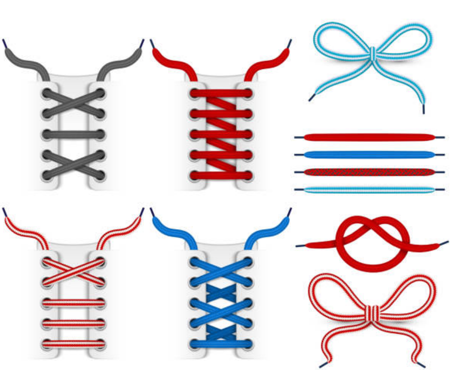
Daniel Tubbenhauer

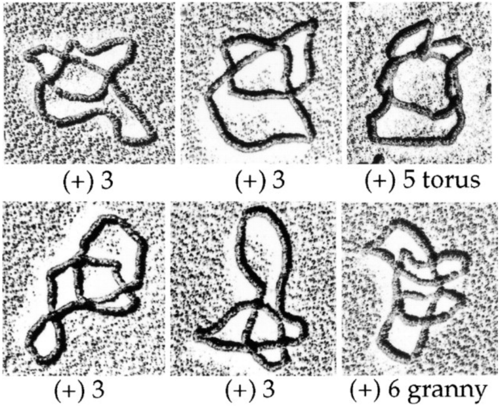


Knot theory

- We all sometimes get stuck within the knots of life
- Since the late 18th century knot theory studies these and other knots
- Knot theory is one of the most appealing and applicable fields of math

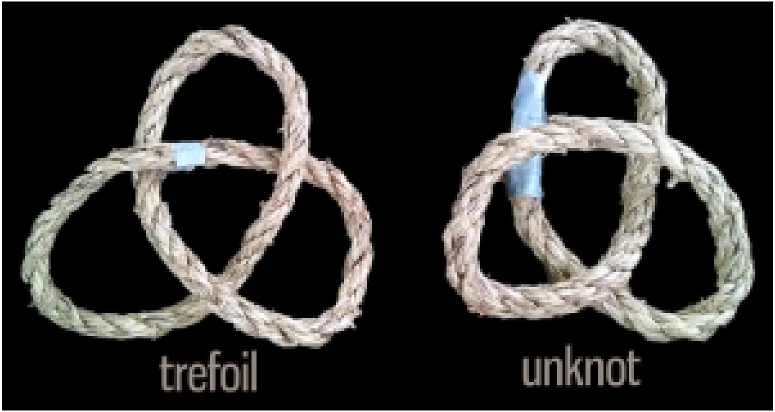
Knot theory





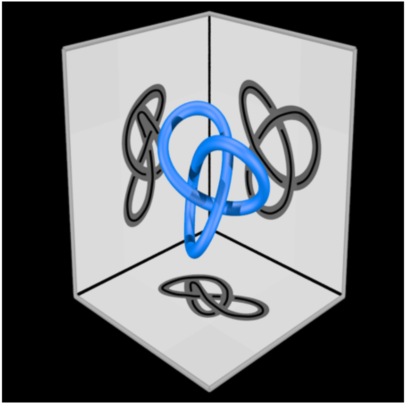
- There are many knots in the real-world: shoelaces, DNA, ...
- Knot theory is the mathematical study of all of these

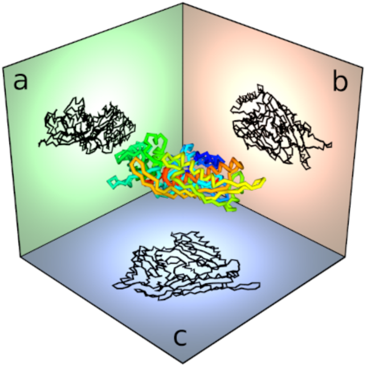
Knot theory



- A mathematical knot is a rope with ends tied together
- That is a necessary because otherwise all knots can be undone
- In practice you can think of your shoelaces tied together

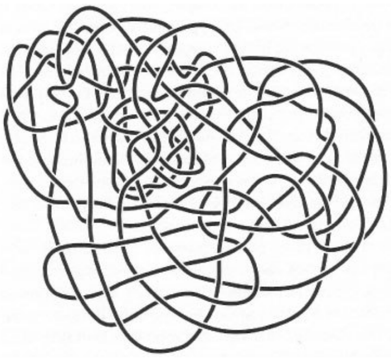
Too many shadows





- Knots are studied via their projections Shadows
- This reduces a 3d problem into a 2d one
- Knot theory deals with the information loss from 3d $\to$ 2d

Too many shadows



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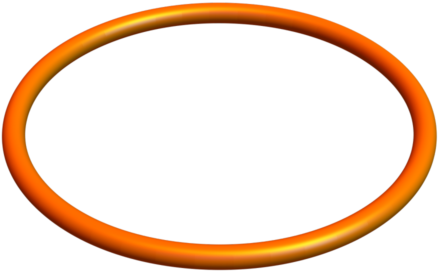
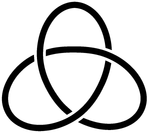
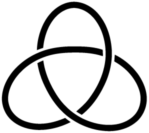
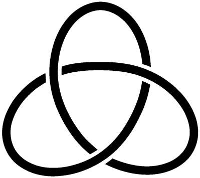
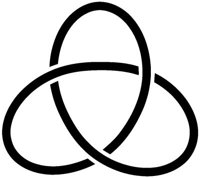


- Problem A knot can be represented by many shadows
- Serious problem Every knot has nasty shadows
- Task Find a way to distinguish knots via their shadows

Enter, quantum algebra

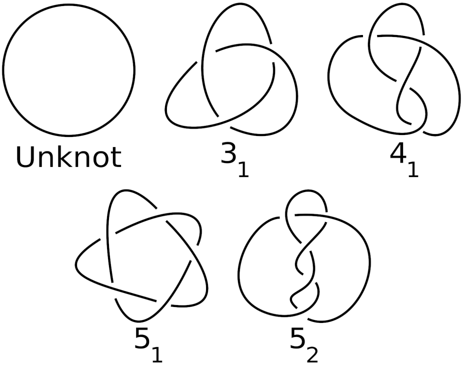
 $\rightsquigarrow q^4 + q^3 + q,$  $\rightsquigarrow q^{-4} + q^{-3} + q^{-1}$

$\Rightarrow$  $\neq$ 

- Knot theory then studies knot invariants
- That is, ones associate an algebraic object (number, polynomial, ...) I_D to a shadow D such that

$$D, D' \text{ present the same knot} \Rightarrow I_D = I_{D'}$$

Enter, quantum algebra



1	3_1	4_1	5_1	5_2
1	q^4+q^3+q	$q^2-q+1-q^{-1}+q^{-2}$	$q^{-2}+q^{-4}-q^{-5}+q^{-6}-q^{-7}$	$q^{-1}-q^{-2}+2q^{-3}-q^{-4}+q^{-5}-q^{-6}$

- Knot invariants are powerful tools to distinguish knots
- But that is (k)not the whole story!

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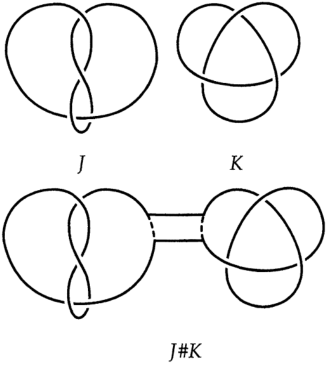


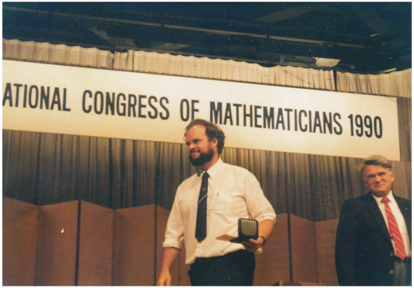
$$J \rightsquigarrow q^2 - q + 1 - q^{-1} + q^{-2}, K \rightsquigarrow q^4 + q^3 + q$$

$$J\#K \rightsquigarrow (q^2 - q + 1 - q^{-1} + q^{-2})(q^4 + q^3 + q)$$

- Another part of the story is that geometry and algebra reflect one another
- Example The geometric operation $\#$ on knots corresponds to polynomial multiplication

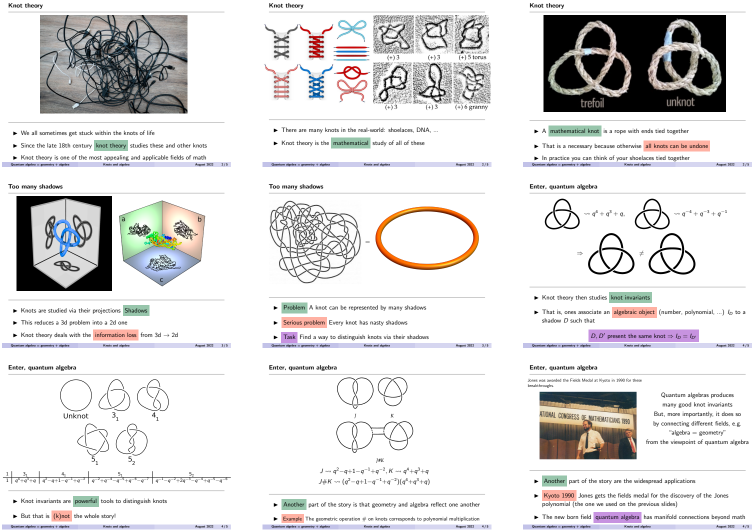
Enter, quantum algebra

Jones was awarded the Fields Medal at Kyoto in 1990 for these breakthroughs.



Quantum algebras produces
many good knot invariants
But, more importantly, it does so
by connecting different fields, e.g.
"algebra = geometry"
from the viewpoint of quantum algebra

- Another part of the story are the widespread applications
- Kyoto 1990 Jones gets the fields medal for the discovery of the Jones polynomial (the one we used on the previous slides)
- The new born field quantum algebra has manifold connections beyond math



There is still much to do...

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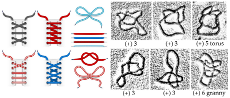
Knot theory



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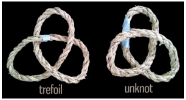

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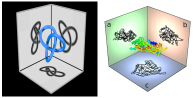

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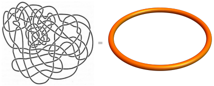

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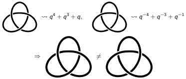

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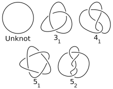

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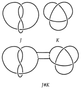

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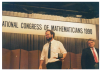

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Thanks for your attention!