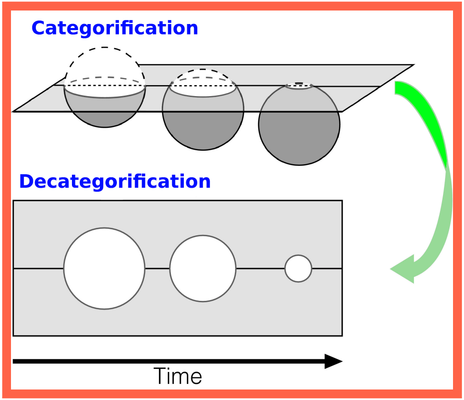
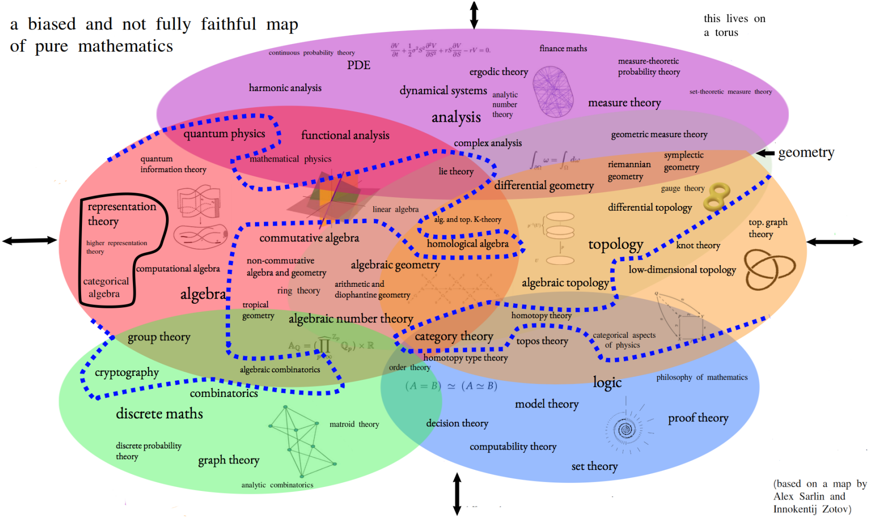
Categorical representation theory and applications

Or: Functors, not matrices

Daniel Tubbenhauer



Where are we?



The six main fields of pure mathematics algebra, analysis, geometry, topology, logic, discrete mathematics

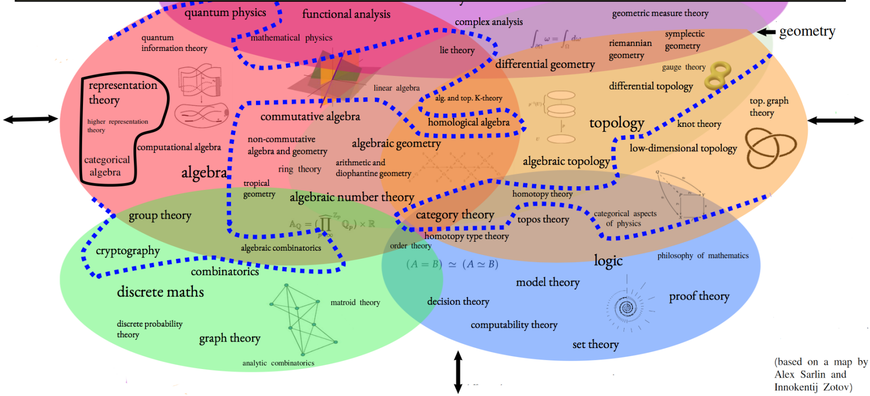
Wh

The map of (pure) mathematics

Black blob. Representation theory and its categorical analog **My research area**

Dashed blob. Where I usually apply them **My research reach**

Applications beyond my current research? **The future (within TUM?)**



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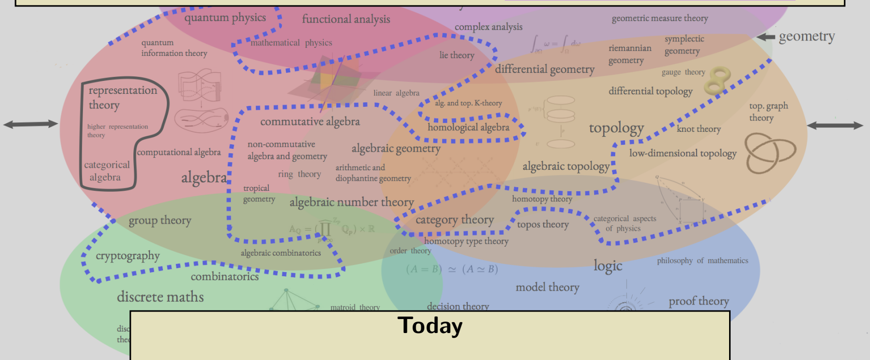


(based on a map by Alex Sarlin and Innokentij Zotov)

Today

(1) I will briefly sketch categorical representation theory

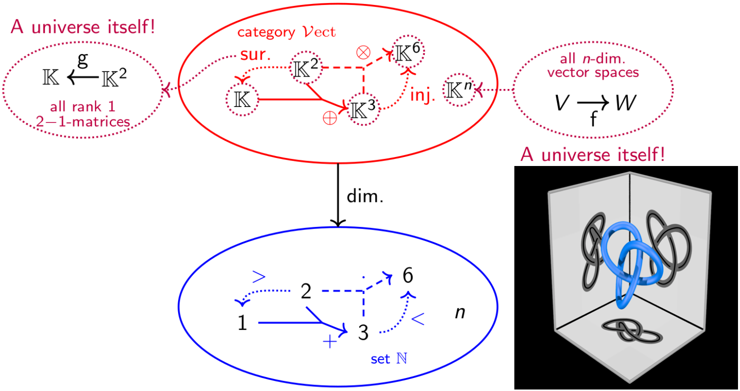
(2) Applications within representation theory

(3) Applications to low dimensional topology

(4) Applications to Lie theory

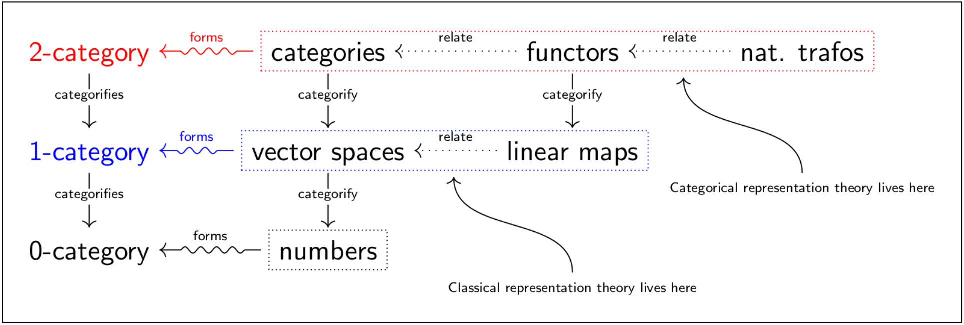
Wh… The six main … etry, topology, lo…

Categorical representation theory = representation theory + categories



- Categorification = replace set-theoretical structures by category-theoretical ones
- Categorifcation reveals hidden structures "Shadow vs. real object"
- Goal Combine categorification and representation theory

Categorical representation theory = representation theory + categories



Rep of a group: group elements $\mapsto$ matrices

Cat rep of a group: group elements $\mapsto$ functors, relations $\mapsto$ nat trafos

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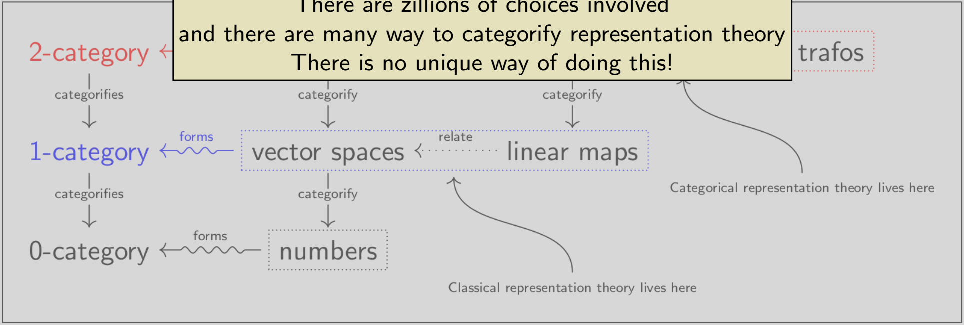
- Representation theory = study of actions of groups/algebras/Lie algebras/etc.
- Cat representation theory = study of actions of (2-)categories

Categorical … ategories

Careful

There are zillions of choices involved
and there are many way to categorify representation theory
There is no unique way of doing this!



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- Representation theory = study of actions of groups/algebras/Lie algebras/etc.
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Categorical … ategories

2-category … trafos

categorifie…

1-categ…

categorifie…

0-categ…

…eory lives here

Cat… …fos

Careful

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and there are many way to categorify representation theory
There is no unique way of doing this!

Some examples of categorical representation theory

Jones, Ocneanu, Popa, others ∼1990
Categorical group rep theory on semisimple categories

Etingof, Ostrik, others ∼2005
Categorical algebra rep theory on abelian categories

Chuang, Rouquier, Khovanov, Lauda, others ∼2005
Categorical Lie algebra representations on additive categories

Joint with Mackaay, Mazorchuk, Miemietz, Zhang, others ∼2011
Categorical algebra rep theory on additive categories

- Representation th… …ebras/Lie algebras/etc.
- Cat representatio… …egories

Today

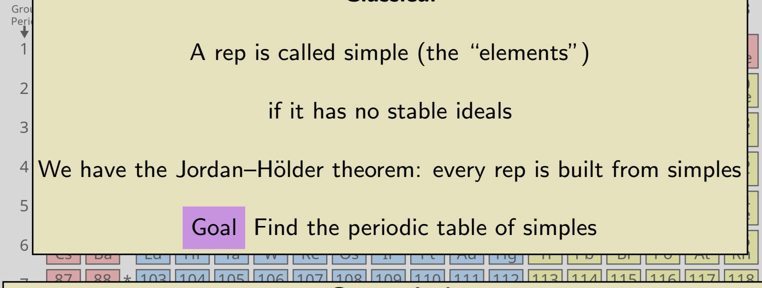
An example instead of formal definitions
and applications of the theory

Instead of G-reps study (G-rep)-reps

Group → / Period ↓	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca		21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr		39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	*	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	**	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
			*	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			**	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

- Goal of chemistry Find the periodic table of elements
- Goal of group theory Find the periodic table of simple groups
- Goal of (cat) rep theory Find the periodic table of simple reps

Instead of G-reps study (G-rep)-reps



Classical

A rep is called simple (the "elements")

if it has no stable ideals

We have the Jordan–Hölder theorem: every rep is built from simples

Goal Find the periodic table of simples

Categorical

A cat rep is called simple (the "elements")

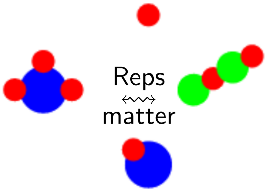
if it has no stable 2-ideals

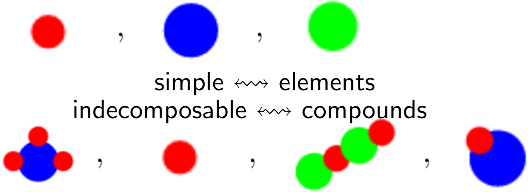
We have the weak Jordan–Hölder theorem: every rep is built from simples

Goal Find the periodic table of simples

- Goal of (cat) rep theory Find the periodic table of simple reps

Instead of G-reps study (G-rep)-reps





- Let $\mathscr{S} = \mathscr{R}\mathrm{ep}(G, \mathbb{C})$
- $\mathscr{S}$ is a 2-cat ✓
- $\mathscr{S}$ is $\mathbb{K}$-linear ✓
- $\mathscr{S}$ is additive ✓
- $\mathscr{S}$ is idempotent complete ✓
- $\mathscr{S}$ has fin dim hom spaces ✓
- $\mathscr{S}$ has finitely many indecomposable objects ✓
- $\mathscr{S}$ has dualities ✓

The actors are finitary/fiat 2-cats
They act on finitary cats

finitary (all properties from "is a 2-cat" to "has finitely many indecomposable objects")

fiat (all properties from "is a 2-cat" to "has dualities")

Instead of G-reps study (G-rep)-reps

Examples instead of formal defs

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ (G a finite group)
- $\mathscr{C}$ is part of the theory. For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation $\mathbb{1}$

- The regular cat representation $\mathscr{M} \colon \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array}$$

- The decategorification is the regular representation

Instead of G-reps study (G-rep)-reps

- Let $K \subset G$ be a subgroup
- $\mathcal{R}\mathrm{ep}(K)$ is a cat representation of $\mathscr{R}\mathrm{ep}(G)$, with action

$$\mathcal{R}es_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}(\mathcal{R}\mathrm{ep}(K)),$$

  which is indeed a cat action because $\mathcal{R}es_K^G$ is a $\otimes$-functor
- The decategorifications are $\mathbb{N}$-representations

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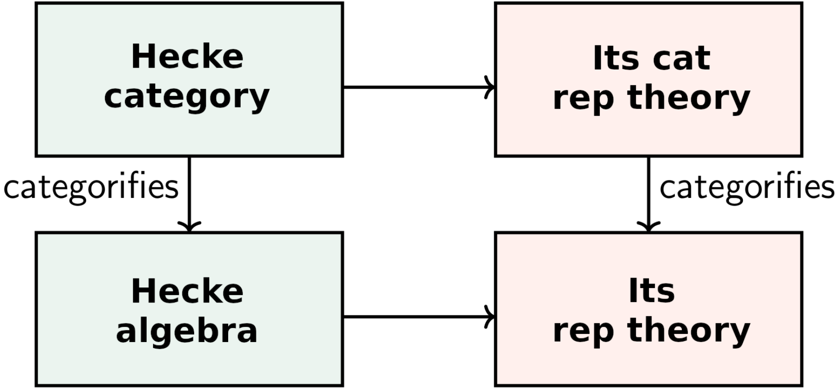
- The decategorifications are $\mathbb{N}$-representations

Theorem

Completeness All simples of $\mathscr{R}\mathrm{ep}(G, \mathbb{C})$ are of the form $\mathcal{V}(K, \mathit{twist})$

Non-redundancy We have $\mathcal{V}(K, \mathit{twist}) \cong \mathcal{V}(K', \mathit{twist}')$
$\Leftrightarrow$
the subgroups and twists are conjugate

Instead of G-reps study (G-rep)-reps



- Symmetric group $\xleftarrow{\textit{generalize}}$ Hecke algebras $\xleftarrow{\textit{categorify}}$ Hecke categories
- We used the $\mathscr{R}\mathrm{ep}(G,\mathbb{C})$-result to classify simple reps of the Hecke categories
- This categorifies rep theory of Hecke algebras

Instea

cat es

- Sy
- W tegories
- T

Advancing the abstract theory over 8 years

$\sim$2011:

Cell 2-representations of finitary 2-categories

Volodymyr Mazorchuk and Vanessa Miemietz

$\sim$2016:

Two-color Soergel Calculus and Simple Transitive 2-representations

Marco Mackaaij and Daniel Tubbenhauer

$\sim$2017:

Simple Transitive 2-Representations via (Co-)Algebra 1-Morphisms

Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz & Daniel Tubbenhauer

$\sim$2019:

Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz, Daniel Tubbenhauer and Xiaoting Zhang*

Finitary birepresentations of finitary bicategories

$\sim$2019:

SIMPLE TRANSITIVE 2-REPRESENTATIONS OF SOERGEL BIMODULES FOR FINITE COXETER TYPES

MARCO MACKAAY, VOLODYMYR MAZORCHUK, VANESSA MIEMIETZ, DANIEL TUBBENHAUER AND XIAOTING ZHANG

Applications of categorical representation theory

From Burnside's book Theory of Groups of Finite Order

first edition ∼1897

> It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

second edition ∼1911

> VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.
>
> In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

- Rep theory is everywhere in math & the sciences but it took a while to get started
- Short and long terms goal Find applications of categorical rep theory

There are many more applications, and not just by me
Here are three examples:

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 1, Pages 203–271
S 0894-0347(01)00374-5
Article electronically published on September 24, 2001

QUIVERS, FLOER COHOMOLOGY, AND BRAID GROUP ACTIONS

MIKHAIL KHOVANOV AND PAUL SEIDEL

Faithful 2-reps of braid groups **Low dim top, symplectic geometry**

Annals of Mathematics, **167** (2008), 245–298

Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification

By JOSEPH CHUANG* and RAPHAËL ROUQUIER

Broué's abelian defect conjecture via 2-reps **Finite groups, rep theory**

Annals of Mathematics **180** (2014), 1089–1136
http://dx.doi.org/10.4007/annals.2014.180.3.6

The Hodge theory of Soergel bimodules

By BEN ELIAS and GEORDIE WILLIAMSON

The Kazhdan–Lusztig conjecture using the Hecke category **Combinatorics, geometry**

Applic

Vol. 101, No. 3 DUKE MATHEMATICAL JOURNAL © 2000

A CATEGORIFICATION OF THE JONES POLYNOMIAL

MIKHAIL KHOVANOV

bedded in $\mathbb{R}^3 \times [0,1]$. In the categorical language, we expect to get a functor from the category of ($\mathbb{Z}_2$-extended) oriented link cobordisms to the category of bigraded R-modules and module homomorphisms.

Functoriality

Euler characteristic $\xleftarrow{\text{categorifies}}$ homologies $\xrightarrow[\text{continuous maps}]{\text{knowns}}$ functoriality

knot polynomials $\xleftarrow{\text{categorifies}}$ knot homologies $\xrightarrow[\text{maps between knots}]{\text{knowns}}$ functoriality

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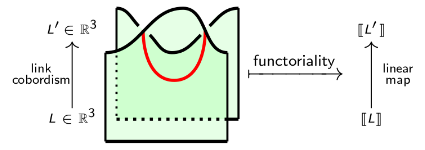
Proc. London Math. Soc. (3) 117 (2018) 996–1040

Functoriality of colored link homologies

Michael Ehrig, Daniel Tubbenhauer and Paul Wedrich



We proved Khovanov's "conjecture" using cat rep theory

Low dim top

- Re ... started
- S

for Z. By semi-infinite Ringel duality [BS18, Section 4], we have the following consequence of Theorem A.

Corollary A. *There is an equivalence of additive, $\mathbb{K}$-linear categories*

$$\mathscr{F}' : \mathbf{Tilt} \xrightarrow{\cong} p\mathbf{Mod}\text{-}\mathrm{Z},$$

sending indecomposable tilting modules to indecomposable projectives.

Its about morphisms!

Transformation Groups ©Springer Science+Business Media New York (2016)
Vol. 22, No. 1, 2017, pp. 29–89

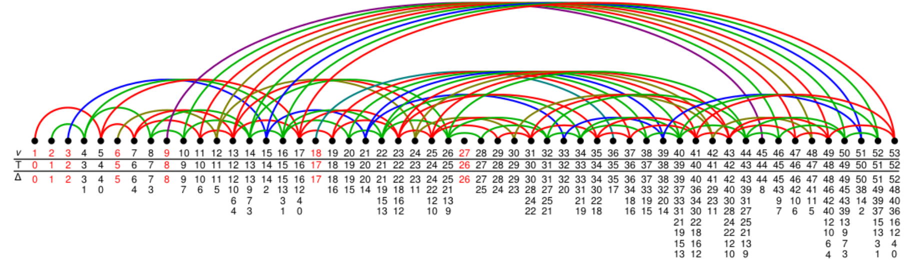
DIAGRAM CATEGORIES FOR U_q-TILTING MODULES AT ROOTS OF UNITY

H. H. ANDERSEN D. TUBBENHAUER

REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 25, Pages 440–480 (June 3, 2021)
https://doi.org/10.1090/ert/569

QUIVERS FOR SL_2 TILTING MODULES

DANIEL TUBBENHAUER AND PAUL WEDRICH



We used cat rep theory of the Hecke category to study modular reps of SL_2 Lie theory

▶ Short and long terms goal Find applications of categorical rep theory

From Burnside's book Theory of Groups of Finite Order

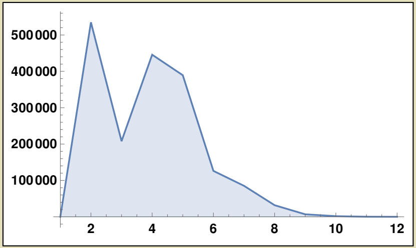
Monoidal cats and their reps are a potential source of Diffie–Hellman-type-protocols

[Submitted on 5 Jan 2022]

Monoidal categories, representation gap and cryptography

Mikhail Khovanov, Maithreya Sitaraman, Daniel Tubbenhauer

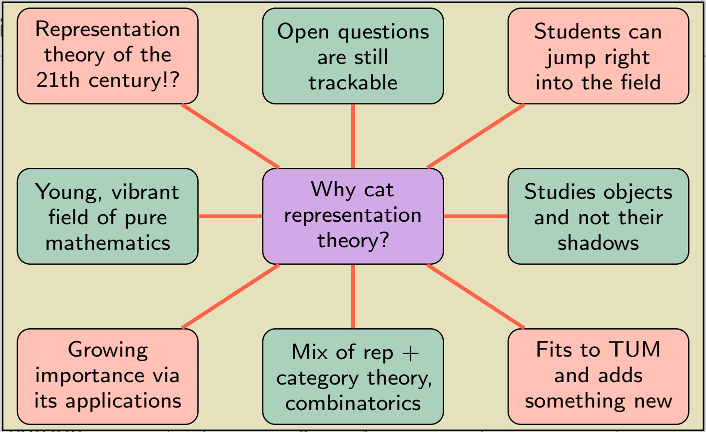
Comments:	45 pages, many figures, comments welcome
Subjects:	**Representation Theory (math.RT)**; Cryptography and Security (cs.CR); Group Theory (math.GR); Quantum Algebra (math.QA)
MSC classes:	18M05 (20M30 94A60)
Cite as:	arXiv:2201.01805 **[math.RT]**
	(or arXiv:2201.01805v1 **[math.RT]** for this version)
	https://doi.org/10.48550/arXiv.2201.01805



We are currently investigating applications to cryptography

▶ Short and long terms goal Find applications of categorical rep theory

Applicati



edition

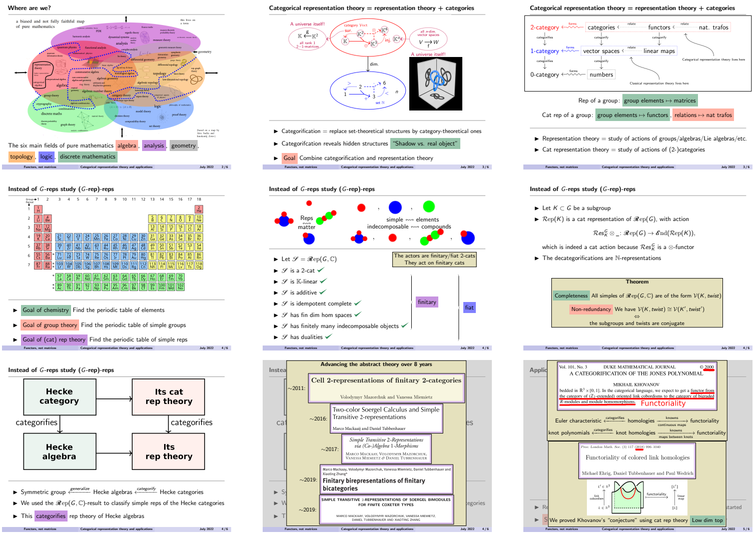
Future at TUM?!

Invariant theory ⇔ cat versions of Schur–Weyl duality ⇔ cat rep theory

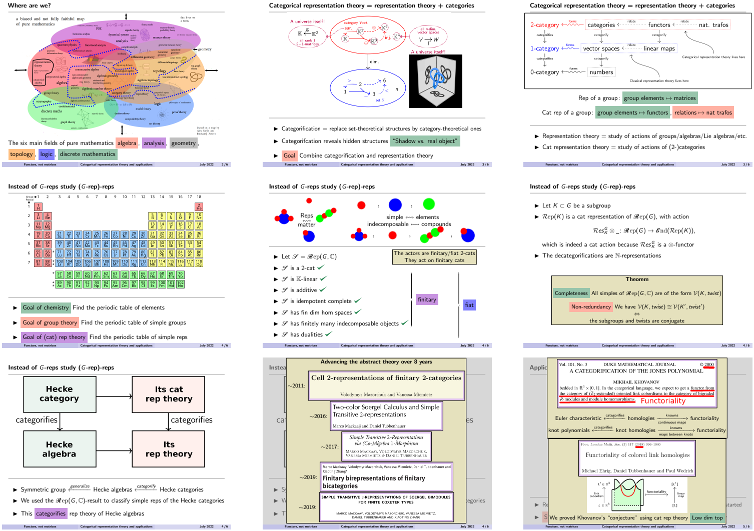
Algebraic geometry in char p ⇔ Satake equivalence ⇔ cat rep theory

Deligne–Lusztig varieties ⇔ cat actions of Kac–Moody algebras ⇔ cat rep theory

Higher categories ⇔ version of TQFTs ⇔ cat rep theory



There is still much to do...



Thanks for your attention!