What is...semisimplicity?

Or: The finite group miracle

S_3 acting on itself

S_3 acts on $\mathbb{C}[S_3]$:

$$
(132) \leftrightsquigarrow
\begin{array}{c}
 \\ id \\ (12) \\ (23) \\ (123) \\ (132) \\ (13)
\end{array}
\begin{array}{c}
\begin{array}{cccccc} id & (12) & (23) & (123) & (132) & (13) \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

- Any group G acts on its group ring $\mathbb{K}[G]$ Action on itself
- $\mathbb{K}[G]$ is called the regular representation
- A miracle happens for $\mathbb{K} = \mathbb{C}$

A simultaneous base change

S_3 acts on $\mathbb{C}[S_3]$:

$$(132) \leftrightsquigarrow \begin{array}{c} \\ id \\ (12) \\ (23) \\ (123) \\ (132) \\ (13) \end{array} \begin{array}{c} \begin{array}{cccccc} id & (12) & (23) & (123) & (132) & (13) \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$(132) \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- After base change, the matrices take block form
- This works simultaneously (above only one of six matrices)

The miracle

S_3 acts on $\mathbb{C}[S_3]$:

$$(12) \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(132) \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- One finds three simple S_3-representations $L_{triv}, L_{stand}, L_{sgn}$
- We have the miraculous formulas

$$\mathbb{C}[S_3] \cong \underbrace{L_{triv}}_{\dim L_{triv}\text{-times}} \oplus \underbrace{L_{stand} \oplus L_{stand}}_{\dim L_{stand}\text{-times}} \oplus \underbrace{L_{sgn}}_{\dim L_{sgn}\text{-times}}$$

$$|S_3| = 1^2 + 2^2 + 1^1 = (\dim L_{triv})^2 + (\dim L_{stand})^2 + (\dim L_{sgn})^2$$

For completeness: A formal statement

G is semisimple over $\mathbb{C}$!

- Every G-representation on a $\mathbb{C}$-vector space V is completely reducible, that is

$$V \cong L_1 \oplus \ldots \oplus L_k$$

  for simple G-representations L_j

- All simple G-representations appear in $\mathbb{C}[G]$ and

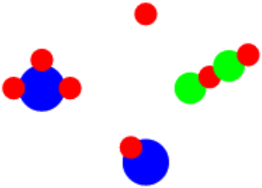
$$\mathbb{C}[G] \cong \bigoplus_{\text{simples}} L^{\oplus \dim L}$$

- We have

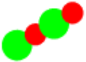
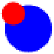
$$|G| = \sum_{\text{simples}} (\dim L)^2$$

- The above is called Maschke's theorem
- It actually works more general, namely for $\mathrm{char}(\mathbb{K}) \nmid |G|$

This is weird and surprising!

representations ⇜⇝ matter 

simples ⇜⇝ elements  ,  , 

indecomposables ⇜⇝ compounds  ,  ,  , 

- Semisimplicity ⇔ "simple=indecomposable"
- Thus, for complex representations of finite groups we have

There are no nontrivial compounds!

Thank you for your attention!

I hope that was of some help.