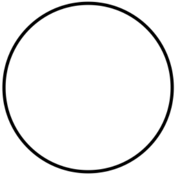
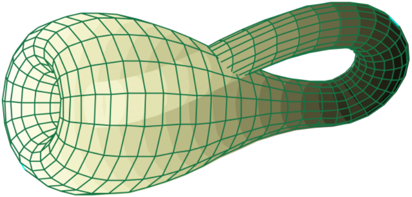
What is...a defect?

Or: Measuring complexity

Numerical measurements



$\stackrel{?}{=}$



- H_* distinguishes S^1 from K

$$H_*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_*(K) \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$$

- P does not distinguish S^1 from K

$$P(S^1) = P(K) = 1 + t$$

- The Poincaré polynomial P and homology H_* are "numerical" invariants of manifolds
- P is easy but not rich, H_* is rich but not easy
- Goal Find analogs in modular rep theory Defect Defect group

Defect

$S_3, \mathbb{C}$:

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

,

$S_3, \overline{\mathbb{F}}_2$:

```
-----------------
Class |   1     3
Size  |   1     2
Order |   1     3
-----------------
p  =  2   1     3
p  =  3   1     1
-----------------
X.1   +   1     1
X.2   +   1 -   1
X.3   +   2    -1
```

```
T := CharacterTable(SymmetricGroup(3));
Blocks(T,2);
```

```
[
    { 1, 2 },
    { 3 }
]
[ 1, 0 ]
```

- Let a such that $|G| = p^a m$ with $p \nmid m$
- Let the defect $d = d(B) \in \mathbb{N}$ be the minimum such that

$$p^{a-d} \mid \chi_\phi(1)$$

for all simples ϕ in a block B

The definition of a defect group is opaque

For $\mathbb{K}$ algebraically closed of characteristic p:

- There is an up to reordering unique decomposition

$$\mathbb{K}[G] \cong \bigoplus_{i=1}^{k} B_i \quad B_i \text{ is an indecomposable two-sided ideal summand}$$

Existence and uniqueness

- Primitive central idempotents $\overset{1:1}{\longleftrightarrow}$ blocks Construction
- The simples are partitioned by the blocks Decomposition of the problem
- Simple characters/$\mathbb{C}$ fit into blocks via simple Brauer characters

Detected: $e_B = \sum_{g \in G} a_g g \mapsto \sum_{g \in C_G(D)} a_g g \neq 0$

A defect group $D = D(B)$ of a block B is:

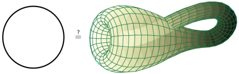
- A p-subgroup of G maximal under inclusion
- subject to the constraint that the block idempotent e_B is detected by D

For completeness: A formal statement

Defect groups exists and are unique up to conjugation **Existence and uniqueness**

(For the defect existence and uniqueness and uniqueness is clear)

They have order p^d where d is the defect



$H_* \xrightarrow{dim/\mathbb{Q}} P$ $\rightsquigarrow$ $D \xrightarrow{size} p^d$ $\mathbb{Z}/p^2\mathbb{Z} \stackrel{?}{=} \mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$

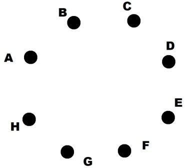
- We can thus say the defect group of B
- For $d = 0$ the defect group is trivial
- For $d = 1$ the defect group is $\mathbb{Z}/p\mathbb{Z}$
- For $d = 2$ the defect group is $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$
- Magma can compute defect groups:

```
T := CharacterTable(SymmetricGroup(3));
Blocks(T,2);
DefectGroup(T[1],2);
```

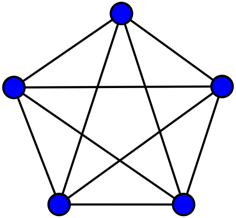
```
Permutation group acting on a set of cardinality 3
Order = 2
    (2, 3)
```

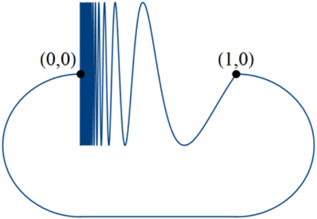
Classification results are hard

dim 0 is easy:



dim 1 is
a bit nasty:





higher dim: well...

- Blocks with trivial defect are matrices and thus are fully classified
- Blocks with cyclic defect are almost fully classified
- Beyond cyclic defect all hell break loose

Thank you for your attention!

I hope that was of some help.