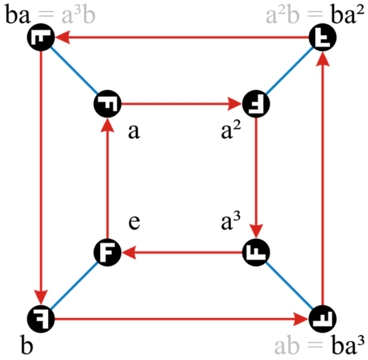
What are...Brauer characters?

Or: Ignore the ground field

Vanishing traces

D_4 acts on D_4



- Every group can act on itself ⇒ regular rep
- The underlying geometric object can be thought of as the Cayley graph
- Huge problem The character of the regular rep is zero if $p \mid |G|$

It gets even worse if $p \mid |G|$

S_4:

```
-----------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
-----------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
-----------------------
X.1    +  1  1  1  1  1
X.2    +  1  1 -1  1 -1
X.3    +  2  2  0 -1  0
X.4    +  3 -1 -1  0  1
X.5    +  3 -1  1  0 -1
```

```
a = matrix(ZZ,[[1,1,1,1,1],[1,1,-1,1,-1],[2,2,0,-1,0],[3,-1,-1,0,1],[3,-1,1,0,-1]]);
b = matrix(GF(5),[[1,1,1,1,1],[1,1,-1,1,-1],[2,2,0,-1,0],[3,-1,-1,0,1],[3,-1,1,0,-1]]);
c = matrix(GF(3),[[1,1,1,1,1],[1,1,-1,1,-1],[2,2,0,-1,0],[3,-1,-1,0,1],[3,-1,1,0,-1]]);
d = matrix(GF(2),[[1,1,1,1,1],[1,1,-1,1,-1],[2,2,0,-1,0],[3,-1,-1,0,1],[3,-1,1,0,-1]]);
print(a.rank())     5
print(b.rank())     5
print(c.rank())     4
print(d.rank())     2
```

- The rep $\phi^{\oplus p}$ has always zero character $\Rightarrow$ characters can not distinguish reps
- Naively specialized simple characters have dependencies $\Rightarrow$ we have the wrong count
- Looks like character theory is dead

Brauer: Let's pretend we have complex numbers

$$\text{permutation } (142)(3): \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
e = matrix(CyclotomicField(3),[[0,1,0,0],[0,0,0,1],[0,0,1,0],[1,0,0,0]]);
f = matrix(GF(2),[[0,1,0,0],[0,0,0,1],[0,0,1,0],[1,0,0,0]]);
g = matrix(GF(3),[[0,1,0,0],[0,0,0,1],[0,0,1,0],[1,0,0,0]]);
print(e.eigenvalues())   [zeta3, -zeta3 - 1, 1, 1]
print(f.eigenvalues())   [1, 1, z2 + 1, z2]
print(g.eigenvalues())   [1, 1, 1, 1]
```

- Trace = sum of eigenvalues
- Eigenvalues are roots of unity (since elements of G have finite order)
- Pretend that these are complex numbers for $p \nmid$ order $\Rightarrow$ Brauer characters

For completeness: A formal statement

For $\mathrm{char}(\mathbb{K}) = p$ the Brauer characters χ_ϕ^p are complex characters of G-reps such that:

(i) They distinguish simple G-reps:

$$\chi_\phi^p = \chi_\psi^p \Leftrightarrow \phi \cong \psi \text{ for } \phi, \psi \text{ simple}$$

(ii) They form a basis of the class functions $G_{reg} \to \mathbb{C}$

In particular, iso classes of simple G-reps = conjugacy classes of p-regular elements

- G_{reg} = p-regular = elements of G whose order is prime to p
- Over $\mathbb{C}$: $\chi_\phi^p = \chi_\psi^p \Leftrightarrow \phi \cong \psi$ for ϕ, ψ arbitrary reps
- Heuristic way to compute the simple Brauer characters:

$D_5/\mathbb{C}$:

		1	2	3	4
Class		1	2	3	4
Size		1	5	2	2
Order		1	2	5	5
p = 2		1	1	4	3
p = 5		1	2	1	1
X.1	+	1	1	1	1
X.2	+	1	-1	1	1
X.3	+	2	0	Z1	Z1#2
X.4	+	2	0	Z1#2	Z1

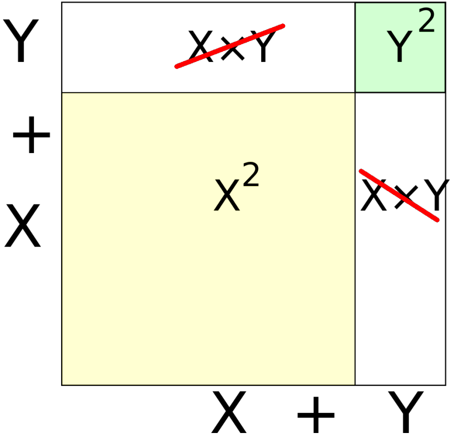
,

$D_5/\overline{\mathbb{F}_2}$:

Class		1	3	4
Size		1	2	2
Order		1	5	5
p = 2		1	4	3
p = 5		1	1	1
X.1	+	1	1	1
X.2	+	1	1	1
X.3	+	2	Z1	Z1#2
X.4	+	2	Z1#2	Z1

Z1=golden ratio

Fresh's dream



- Over $\mathbb{F}_p$: $(X+Y)^p = X^p + Y^p$ so $(X+1)^p = X^p + 1$
- $\Rightarrow$ there are no primitive pth roots of unity over $\mathbb{F}_p$
- Brauer We need to get rid of these anyway but everything else is ok!

Thank you for your attention!

I hope that was of some help.