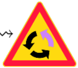
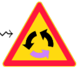
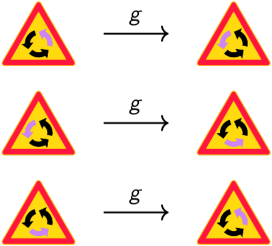
What is...modular representation theory?

Or: Division made hard

Jordan decomposition

$\mathbb{Z}/3\mathbb{Z}$ acts on $\mathbb{C}$ $\Big\{ (1,0,0) \leftrightsquigarrow$  $, (0,1,0) \leftrightsquigarrow$  $, (0,0,1) \leftrightsquigarrow$  $\Big\}$



$$g \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- Apply Jordan decomposition over $\mathbb{C}$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \quad \zeta^2 + \zeta + 1 = 0$$

- Jordan decomposition over $\mathbb{C} \Rightarrow$ regular $\mathbb{Z}/3\mathbb{Z}$ rep decomposes nicely

Let us try Jordan decomposition again

```
a = matrix(CyclotomicField(3),[[0,0,1],[1,0,0],[0,1,0]]);
b = matrix(GF(3),[[0,0,1],[1,0,0],[0,1,0]]);
print(a.jordan_form())
print(b.jordan_form())
```

```
[          1|          0|          0]
[-----------+-----------+-----------]
[          0|      zeta3|          0]
[-----------+-----------+-----------]
[          0|          0|-zeta3 - 1]
[1 1 0]
[0 1 1]
[0 0 1]
```

- Apply Jordan decomposition over $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- Jordan decomposition over $\mathbb{F}_3 \Rightarrow$ regular $\mathbb{Z}/3\mathbb{Z}$ rep is indecomposable

Fixed vectors

$v = (1, 1, 1) =$  $+$  $+$ 

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v = v$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} v = v$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} v = v$$

- v is an eigenvector of the action independent of the field
- $\Rightarrow$ The regular $\mathbb{Z}/3\mathbb{Z}$ rep is not simple over $\mathbb{F}_3$
- $\Rightarrow$ Rep theory of $\mathbb{Z}/3\mathbb{Z}$ over $\mathbb{F}_3$ is not semisimple

For completeness: A formal statement

For finite characteristic things get a bit nasty:

(i) G has a semisimple rep theory over $\overline{\mathbb{F}_p}$ if and only if $p \nmid |G|$ (Maschke's theorem)

(ii) Character theory falls apart:

$$p \nmid |G|: \chi_\phi = \chi_\psi \Leftrightarrow \phi \cong \psi \qquad p \mid |G|: \chi_\phi = \chi_\psi \not\Rightarrow \phi \cong \psi$$

(iii) The notion of a character is fishy for $p \mid |G| \Rightarrow$ need Brauer characters

(iv) Many more crappy things happen

- Over $\mathbb{F}_p$ for $p \neq 3$ everything is fine for $\mathbb{Z}/3\mathbb{Z}$:

```
c = matrix(GF(4),[[0,0,1],[1,0,0],[0,1,0]]);
d = matrix(GF(7),[[0,0,1],[1,0,0],[0,1,0]]);
print(c.jordan_form())
print(d.jordan_form())
```

```
[     1|     0|     0]
[------+------+------]
[     0|    z2|     0]
[------+------+------]
[     0|     0|z2 + 1]
[4|0|0]
[-+-+-]
[0|2|0]
[-+-+-]
[0|0|1]
```

- Characters of $\mathbb{Z}/3\mathbb{Z}$ over $\mathbb{F}_3$:

$$\chi_{\text{triv}\oplus\text{triv}\oplus\text{triv}} = 0 = \chi_{\text{reg rep}} \quad \text{but} \quad \text{triv} \oplus \text{triv} \oplus \text{triv} \ncong \text{reg rep}$$

Eigenvalues of pseudo idempotents

$v = (1,1,1) =$  $+$  $+$ 

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

```
e = matrix(ZZ,[[1,1,1],[1,1,1],[1,1,1]]);
e.eigenvalues()
```

```
[3, 0, 0]
```

- Maschke's theorem uses the total pseudo idempotent $T = \sum_{g \in G} g$
- The leading eigenvalue of T is $|G|$
- Problem We can not divide by $|G|$ if $p \mid |G|$

Thank you for your attention!

I hope that was of some help.