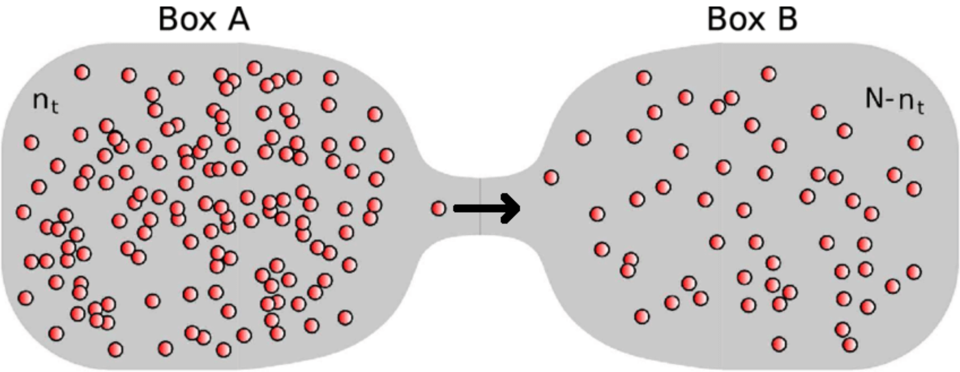
What is...the Ehrenfest model?

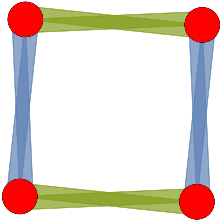
Or: Rep theory and diffusion

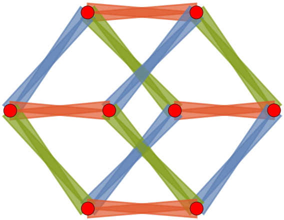
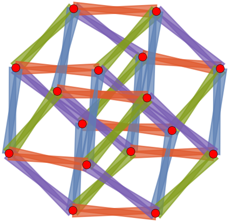
Jumping between boxes



- We have two boxes A and B containing a total of N balls
- At each step, one ball is chosen at random and moved to the other box
- "Random" = all balls are equal
- Question If we start with all N balls in A, what will happen in the long run?

Cayley graphs of $(\mathbb{Z}/2\mathbb{Z})^N$

$N = 1$:  , $N = 2$: 

$N = 3$:  , $N = 4$: 

- Take the standard gen set of $(\mathbb{Z}/2\mathbb{Z})^N$ given by $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ...$
- The Cayley graph of $(\mathbb{Z}/2\mathbb{Z})^N$ is then the Nd cube

Diffusion is (often) a random walk

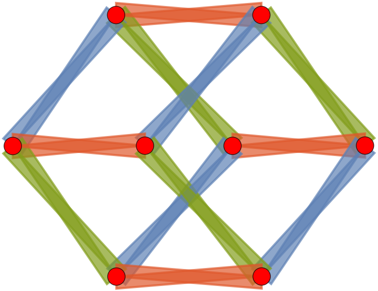
- Diffusion "=" open a perfume bottle in the corner of a room and smell what happens
- Diffusion "=" random walk + the Ehrenfest model is a diffusion model
- The Ehrenfest model "=" random walk (equal prob.) on the Cayley graph of $(\mathbb{Z}/2\mathbb{Z})^N$

For completeness: A formal statement

For the Cayley graph Γ for gen set S of a finite abelian group we have:

- The simple characters/$\mathbb{C}$ are in 1:1 correspondence with eigenvalues of Γ
- The eigenvalues are given by

$$EV_k = \sum_{g \in S} \chi_k(g)$$

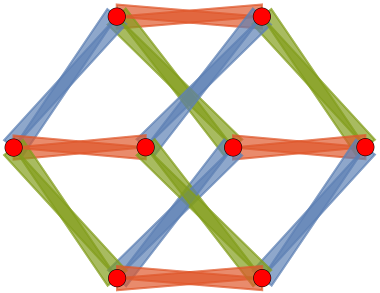


-3,-1,-1,-1,1,1,1,3

```
--------------------------------
Class |   1  2  3  4  5  6  7  8
Size  |   1  1  1  1  1  1  1  1
Order |   1  2  2  2  2  2  2  2
--------------------------------
p  =  2   1  1  1  1  1  1  1  1
--------------------------------
X.1   +   1  1  1  1  1  1  1  1
X.2   +   1 -1 -1  1 -1  1  1 -1
X.3   +   1 -1  1 -1  1 -1  1 -1
X.4   +   1  1 -1 -1 -1 -1  1  1
X.5   +   1  1 -1 -1  1  1 -1 -1
X.6   +   1 -1  1 -1 -1  1 -1  1
X.7   +   1 -1 -1  1  1 -1 -1  1
X.8   +   1  1  1  1 -1 -1 -1 -1
```

S from 6,7,8

Back to Ehrenfest!



$$\xrightarrow[\text{prob}]{\text{scale by}} \begin{pmatrix} 0 & 0.33 & 0.33 & 0 & 0.33 & 0 & 0 & 0 \\ 0.33 & 0 & 0 & 0.33 & 0 & 0.33 & 0 & 0 \\ 0.33 & 0 & 0 & 0.33 & 0 & 0 & 0.33 & 0 \\ 0 & 0.33 & 0.33 & 0 & 0 & 0 & 0 & 0.33 \\ 0.33 & 0 & 0 & 0 & 0 & 0.33 & 0.33 & 0 \\ 0 & 0.33 & 0 & 0 & 0.33 & 0 & 0 & 0.33 \\ 0 & 0 & 0.33 & 0 & 0.33 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0.33 & 0 & 0.33 & 0.33 & 0 \end{pmatrix}$$

$$\text{15th power}: \begin{pmatrix} 0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0.25 \\ 0.25 & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \\ 0.25 & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \\ 0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0.25 \\ 0.25 & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \\ 0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0.25 \\ 0. & 0.25 & 0.25 & 0. & 0.25 & 0. & 0. & 0.25 \\ 0.25 & 0. & 0. & 0.25 & 0. & 0.25 & 0.25 & 0. \end{pmatrix}$$

- Scaling of the adjacency matrix of Γ by prob gives a prob matrix
- The eigenvalues of this scaling determine the long term behavior
- The simple chars encode the long term behavior of the Ehrenfest diffusion

Thank you for your attention!

I hope that was of some help.