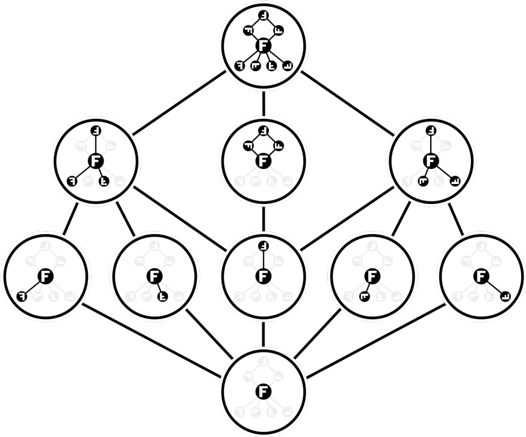
What are...induction and restriction?

Or: How to find lost information! Kind of...

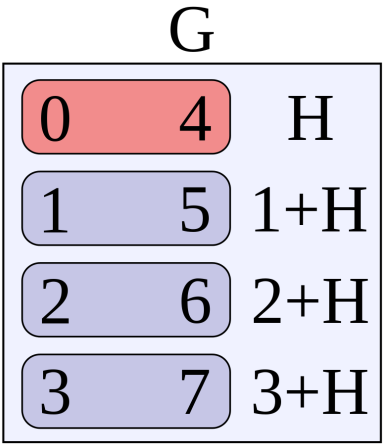
Forgetting is easy



- $\phi\colon G \to \mathrm{GL}(V)$, $H \subset G$ subgroup $\Rightarrow$ restricted rep $\mathrm{Res}^H_G(\phi)\colon H \to \mathrm{GL}(V)$
- $\mathrm{Res}^H_G(\phi)$ is obtained from ϕ by forgetting matrices
- Is there an “inverse” process?

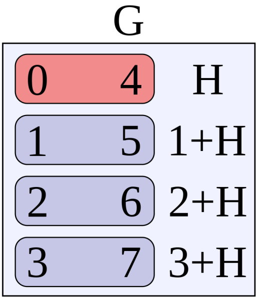
Recover lost information is tricky

$\{0, 4\} \subset \mathbb{Z}/8\mathbb{Z}$:



- Say we look at cosets of $\{0, 4\} \cong \mathbb{Z}/2\mathbb{Z} = H$ in $\mathbb{Z}/8\mathbb{Z} = G$
- A representation ϕ of H contains 25% of the information $[G : H] = 4$
- Idea Make ϕ 4 times as big and force the values for $1+H$, $2+H$, $3+H$ to work

$\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/8\mathbb{Z}$



Say we have $0 \mapsto 1$, $4 \mapsto -1$. Where should we send 1? It should be

$$1 \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} \text{first row} &: -0+1+i \\ \text{second row} &: -1+1+i \\ \text{third row} &: -2+1+i \\ \text{fourth row} &: -3+1+i \end{aligned}$$

Why? Because this forces the correct behavior!

For completeness: A formal definition

Let $H \subset G$ be a subgroup, ϕ a G rep, ψ an H rep

- The restriction $\mathrm{Res}_G^H(\phi)$ is obtained by restricting the action map
- The induction $\mathrm{Ind}_H^G(\psi)$ is obtained by

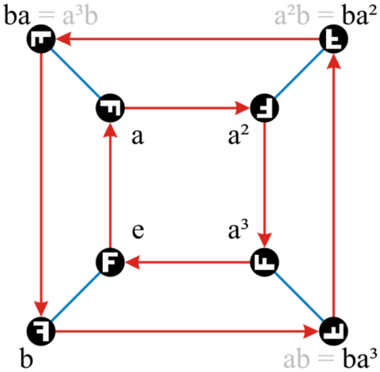
$$\varphi_g^G = \begin{bmatrix} \dot{\varphi}_{t_1^{-1}gt_1} & \dot{\varphi}_{t_1^{-1}gt_2} & \cdots & \dot{\varphi}_{t_1^{-1}gt_m} \\ \dot{\varphi}_{t_2^{-1}gt_1} & \dot{\varphi}_{t_2^{-1}gt_2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \dot{\varphi}_{t_{m-1}^{-1}gt_m} \\ \dot{\varphi}_{t_m^{-1}gt_1} & \cdots & \dot{\varphi}_{t_m^{-1}gt_{m-1}} & \dot{\varphi}_{t_m^{-1}gt_m} \end{bmatrix}$$

where $t_1, \ldots, t_m$ are coset representatives and

$$\dot{\varphi}_x = \begin{cases} \varphi_x & x \in H \\ 0 & x \notin H \end{cases}$$

Example

D_4 acts on D_4



- The regular representation is $\mathrm{Ind}_1^G(1)$
- Why? Well, $t_i^{-1}gt_j = 1$ implies $gt_j = t_i$ which is the action on itself

Thank you for your attention!

I hope that was of some help.