What is...Burnside's theorem?

Or: Groups and representations

Burnside's book Theory of Groups of Finite Order

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

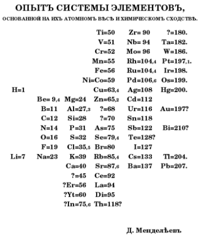
In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

- Top: first edition ∼1897 Burnside: "Rep theory is useless"
- Bottom: second edition ∼1911 Burnside: "Rep theory is amazing"
- In between was Burnside's rep theory proof of Burnside's theorem

Simple groups, simple representations

Chemistry	Group theory	Rep theory
Matter	Groups	Reps
Elements	Simple groups	Simple reps
Simpler substances	Jordan–Hölder theorem	Jordan–Hölder theorem
Periodic table	Classification of simple groups	Classification of simple reps

- Question What are the simplest possible representations?
- Whatever these are, they should play the role of elements in rep theory!



ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,

ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.

			Ti=50	Zr= 90	?=180.
			V=51	Nb= 94	Ta=182.
			Cr=52	Mo= 96	W=186.
			Mn=55	Rh=104,4	Pt=197,1.
			Fe=56	Ru=104,4	Ir=198.
			Ni=Co=59	Pd=106,6	Os=199.
H=1			Cu=63,4	Ag=108	Hg=200.
	Be= 9,4	Mg=24	Zn=65,2	Cd=112	
	B=11	Al=27,3	?=68	Ur=116	Au=197?
	C=12	Si=28	?=70	Sn=118	
	N=14	P=31	As=75	Sb=122	Bi=210?
	O=16	S=32	Se=79,4	Te=128?	
	F=19	Cl=35,5	Br=80	I=127	
Li=7	Na=23	K=39	Rb=85,4	Cs=133	Tl=204.
		Ca=40	Sr=87,6	Ba=137	Pb=207.
		?=45	Ce=92		
		?Er=56	La=94		
		?Yt=60	Di=95		
		?In=75,6	Th=118?		

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- The elements of finite group theory are the finite simple groups
- It is easy to see that G with $|G| = p^k$ is only simple if $G = \mathbb{Z}/p\mathbb{Z}$
- Question What about more than one prime factors?

List of finite simple groups

Order	Factored order	Group
60	$2^2 \cdot 3 \cdot 5$	$A_5 = A_1(4) = A_1(5)$
168	$2^3 \cdot 3 \cdot 7$	$A_1(7) = A_2(2)$
360	$2^3 \cdot 3^2 \cdot 5$	$A_6 = A_1(9) = B_2(2)'$
504	$2^3 \cdot 3^2 \cdot 7$	$A_1(8) = {}^2G_2(3)'$
660	$2^2 \cdot 3 \cdot 5 \cdot 11$	$A_1(11)$
1092	$2^2 \cdot 3 \cdot 7 \cdot 13$	$A_1(13)$
2448	$2^4 \cdot 3^2 \cdot 17$	$A_1(17)$
2520	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	A_7
3420	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	$A_1(19)$
4080	$2^4 \cdot 3 \cdot 5 \cdot 17$	$A_1(16)$
5616	$2^4 \cdot 3^3 \cdot 13$	$A_2(3)$
6048	$2^5 \cdot 3^3 \cdot 7$	${}^2A_2(9) = G_2(2)'$
6072	$2^3 \cdot 3 \cdot 11 \cdot 23$	$A_1(23)$
7800	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$A_1(25)$
7920	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$\mathbf{M_{11}}$

- The first non-abelian group on this list is A_5 with $|A_5| = 2^2 \cdot 3 \cdot 5$
- All other non-abelian groups on this list have at least three prime factors
- Conjecture All non-abelian finite simple groups have at least three prime factors

For completeness: A formal statement

Burnside's theorem Groups of order $p^a q^b$ are solvable

- p, q prime numbers, $a, b \in \mathbb{N}$
- solvable $=$ $\exists$ subnormal series whose quotient groups are abelian
- non-abelian $+$ simple $\Rightarrow$ not solvable
- **Corollary** Groups of order $p^a q^b$ are not simple
- Burnside's proof uses rep theory; it took $\approx$70 years to find a proof without rep theory

On Groups of Order $p^\alpha q^\beta$

W. Burnside

From 1904

First published: 1904 | https://doi.org/10.1112/plms/s2-1.1.388

Published: October 1970

A group theoretic proof of the $p^a q^b$ theorem for odd primes

David M. Goldschmidt

From 1970

A proof sketch for the $p^a q^b$ theorem

- Assume the following slightly technical theorem:

The next theorem is of a somewhat technical nature (meaning that I do not see how to motivate it), but is crucial to proving Burnside's theorem.

Theorem 6.3.9. *Let G be a group of order n and let C be a conjugacy class of G. Suppose that $\varphi\colon G \longrightarrow GL_d(\mathbb{C})$ is an irreducible representation and assume that $h = |C|$ is relatively prime to d. Then either:*

1. *there exists $\lambda \in \mathbb{C}^*$ such that $\varphi_g = \lambda I$ for all $g \in C$; or*
2. *$\chi_\varphi(g) = 0$ for all $g \in C$.*

- The above slightly technical theorem implies

Lemma 6.3.10. *Let G be a finite non-abelian group. Suppose that there is a conjugacy class $C \neq \{1\}$ of G such that $|C| = p^t$ with p prime, $t \geq 0$. Then G is not simple.*

- Assume $p \neq q$ and $a > 0, b > 0$ (the other cases are boring)
- Then G has a subgroup of order q^b
- The existence of this subgroup implies the existence of a conjugacy class of order p^t for some $t \geq 0$
- Done by Lemma 6.3.10

Thank you for your attention!

I hope that was of some help.