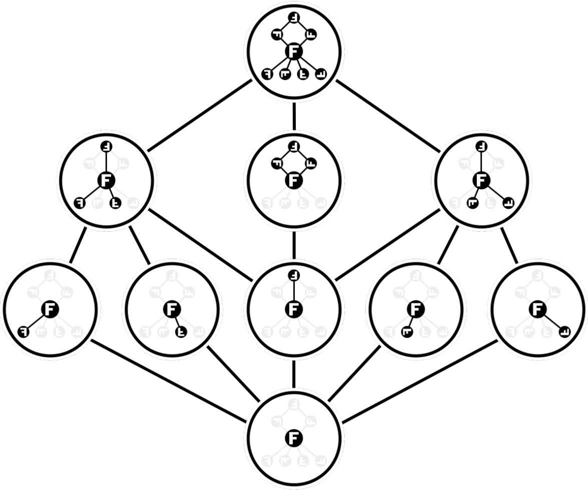
What is...Lagrange's theorem for representations?

Or: Divisibility

Lagrange for groups



- The order of a subgroup of G divides the order of G
- Question What is the analog in the world of representations?

Lagrange for representations?

$|D_4| = 8$ and

```
--------------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
--------------------------
p  =  2   1  1  1  1  2
--------------------------
X.1    +   1  1  1  1  1
X.2    +   1  1 -1  1 -1
X.3    +   1  1  1 -1 -1
X.4    +   1  1 -1 -1  1
X.5    +   2 -2  0  0  0
```

- The dimension of simple D_4-reps divide the order of D_4
- Well, come on, this might be a coincidence

Lagrange for representations!

$|\mathrm{SL}_2(\mathbb{F}_7)| = 2^4 \cdot 3 \cdot 7$ and

```
------------------------------------------------------------
Class |   1  2  3  4  5    6    7    8    9   10   11
Size  |   1  1 56 42 56   24   24   42   42   24   24
Order |   1  2  3  4  6    7    7    8    8   14   14
------------------------------------------------------------
p  =  2   1  1  3  2  3    6    7    4    4    7    6
p  =  3   1  2  1  4  2    7    6    9    8   11   10
p  =  7   1  2  3  4  5    1    1    8    9    2    2
------------------------------------------------------------
X.1   +   1  1  1  1  1    1    1    1    1    1    1
X.2   0   3  3  0 -1  0   Z1 Z1#3    1    1 Z1#3   Z1
X.3   0   3  3  0 -1  0 Z1#3   Z1    1    1   Z1 Z1#3
X.4   0   4 -4  1  0 -1  -Z1-Z1#3    0    0 Z1#3   Z1
X.5   0   4 -4  1  0 -1-Z1#3  -Z1    0    0   Z1 Z1#3
X.6   +   6  6  0  2  0   -1   -1    0    0   -1   -1
X.7   -   6 -6  0  0  0   -1   -1   Z2  -Z2    1    1
X.8   -   6 -6  0  0  0   -1   -1  -Z2   Z2    1    1
X.9   +   7  7  1 -1  1    0    0   -1   -1    0    0
X.10  +   8  8 -1  0 -1    1    1    0    0    1    1
X.11  -   8 -8 -1  0  1    1    1    0    0   -1   -1
```

- The dimension of simple $\mathrm{SL}_2(\mathbb{F}_7)$-reps divide the order of $\mathrm{SL}_2(\mathbb{F}_7)$
- Ok, this is not a coincidence

For completeness: A formal statement

The dimension of simple G-reps divides $|G|$ **Lagrange**

- The dimension of simple G-reps even divides $[G : Z(G)]$
- The number of 1d simple G-reps also divides $|G|$ as well

$|D_4| = 8, |Z(D_4)| = 2$ and

```
-----------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
-----------------------
p  =  2   1  1  1  1  2
-----------------------
X.1    +   1  1  1  1  1
X.2    +   1  1 -1  1 -1
X.3    +   1  1  1 -1 -1
X.4    +   1  1 -1 -1  1
X.5    +   2 -2  0  0  0
```

Some consequences

Corollary 6.2.6. *Let p be a prime and let G be a group of order p^2. Then G is ~~an~~ abelian.*

Corollary 6.2.8. *Let p, q be primes with $p < q$ and $q \not\equiv 1 \bmod p$. Then any group G of order pq is abelian.*

The above are examples of consequences in group theory
Proof? Always along the same lines:

Proof. Let $d_1, \ldots, d_s$ be the degrees of the irreducible representations of G. Then d_i can be 1, p or p^2. Since the trivial representation has degree 1 and

$$p^2 = |G| = d_1^2 + \cdots + d_s^2$$

it follows that all $d_i = 1$ and hence G is abelian. □

Thank you for your attention!

I hope that was of some help.