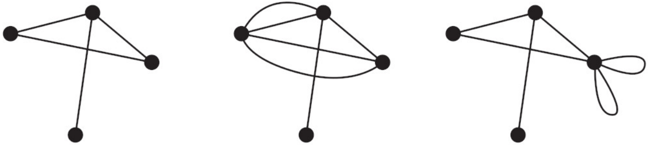
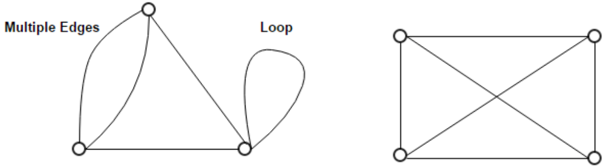
What is...the Erdős–Gabbai theorem?

Or: Realizing graphs

Simple graphs



simple graph — *nonsimple graph with multiple edges* — *nonsimple graph with loops*

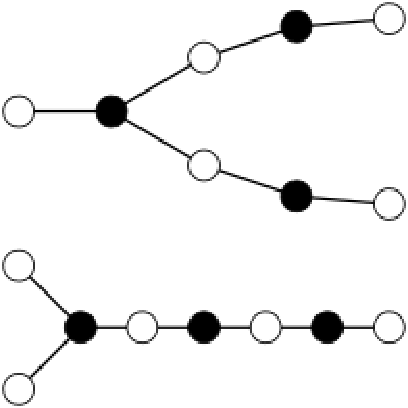


Not a Simple Graph — **Simple Graph**

- A graph without loops or multiple edges is called simple
- Question How much information determines whether simple graphs exist?

Degree sequences

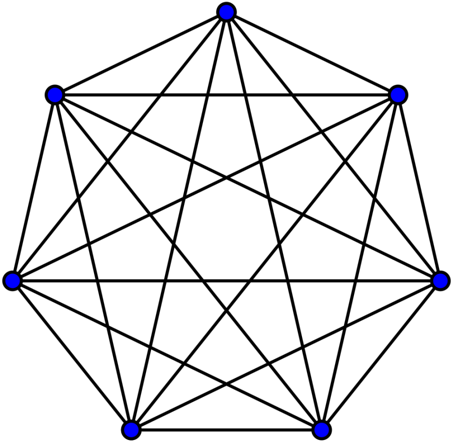
$3 \geq 2 \geq 2 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1$:



- Degree of a vertex = number of its neighbors
- $d_1 \geq ... \geq d_n$ are called graphic if $\exists$ a simple graph with these degrees
- Question Can we decide whether a given $d_1 \geq ... \geq d_n$ is graphic?

A first upper bound

$6 \geq 6 \geq 6 \geq 6 \geq 6 \geq 6 \geq 6 \geq 6$:



- We cannot put more edges to complete graphs K_n
- In particular, $d_1 \geq ... \geq d_n$ graphic implies $d_1 + ... + d_n \leq n(n-1)$
- It is unclear whether one gets better conditions

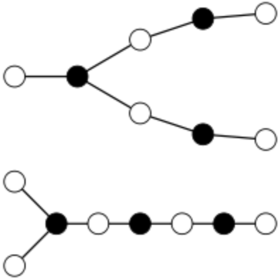
Enter, the theorem

$d_1 \geq \ldots \geq d_n$ is graphic **if and only if**

- $d_1 + \ldots + d_n$ is even **Handshaking condition**
- $d_1 + \ldots + d_k \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ **Complete graph condition**

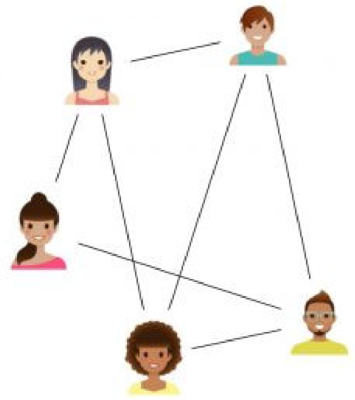
- There are many graphs with a given graphic sequence, *e.g.*:

$3 \geq 2 \geq 2 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1$:



- The theorem is also not constructive
- There are generalizations beyond the class of simple graphs

Handshaking condition



- The number of handshakes is always even:

$$\sum_v deg(v) = 2|E|$$

  This was known to Euler and is easy to prove
- The main point is the complete graph condition

Thank you for your attention!

I hope that was of some help.