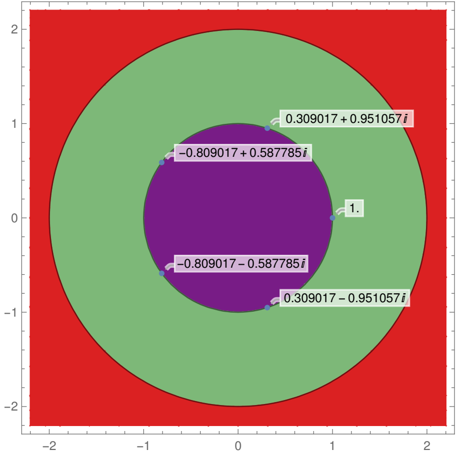
What is...Kronecker's theorem?

Or: Darts and polynomials

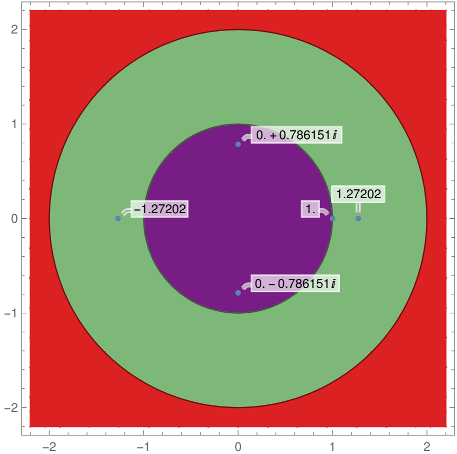
Playing with polynomials



Roots of $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$

- Random polynomials have random roots Not much more to say
- Kronecker Maybe we can say something if we only have “small” roots?

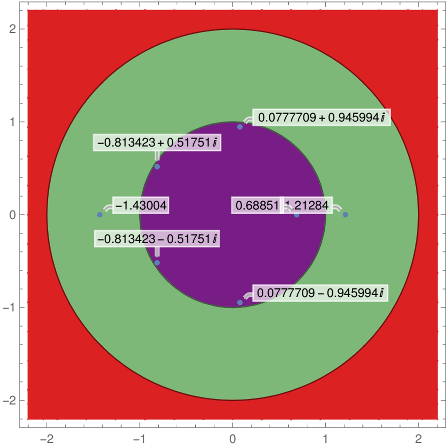
Trying to get roots into the center



Roots of $x^5 - x^4 - x^3 + x^2 - x + 1$

- Factoring over $\mathbb{Z}$ gives $x^5 - x^4 - x^3 + x^2 - x + 1 = (x-1)(x^4 - x^2 - 1)$
- The first factor is a cyclotomic polynomial, the second is not
- The first factor has roots in the unit circle, the second does not

A few more attempts later



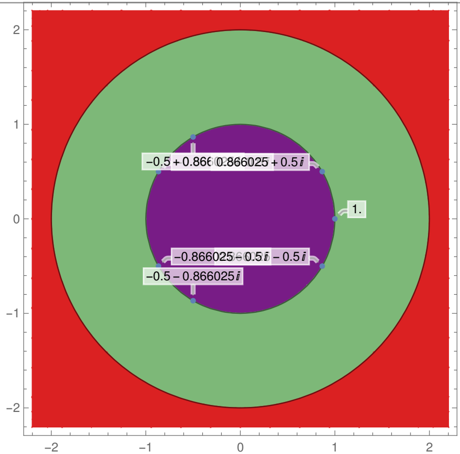
Roots of $x^7 + x^6 - x^5 - x^4 - x^3 - x^2 + 1$

- $x^7 + x^6 - x^5 - x^4 - x^3 - x^2 + 1$ is irreducible over $\mathbb{Z}$
- The roots are not in the unit circle
- Maybe there are no irreducible $\mathbb{Z}$ polys whose roots are strictly in the unit circle?

Enter, the theorem

Let $q \in \mathbb{C}$ nonzero be an algebraic integer such that all its algebraic conjugates have absolute values ≤ 1

Then the abs value of q is $= 1$



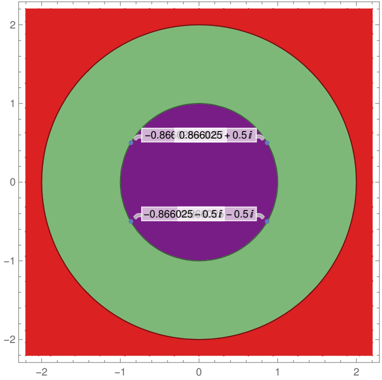
Roots of $x^7 - x^5 - x^4 + x^3 + x^2 - 1 = (x+1)(x^2+x+1)(x^4-x^2+1)$

Cyclotomic polynomials

$$\Phi_1(x) = x - 1$$
$$\Phi_2(x) = x + 1$$
$$\Phi_3(x) = x^2 + x + 1$$
$$\Phi_4(x) = x^2 + 1$$
$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$$
$$\Phi_6(x) = x^2 - x + 1$$

$$\Phi_6(x) = x^2 - x + 1$$
$$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$
$$\Phi_8(x) = x^4 + 1$$
$$\Phi_9(x) = x^6 + x^3 + 1$$
$$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$$
$$\Phi_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$
$$\Phi_{12}(x) = x^4 - x^2 + 1$$

$x^4 - x^2 + 1$:



The polynomials realizing the q are the cyclotomic polynomials

Thank you for your attention!

I hope that was of some help.