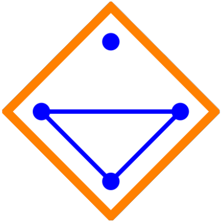
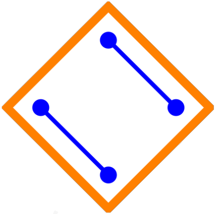
What are...set partitions?

Or: Prototypical lattices!?

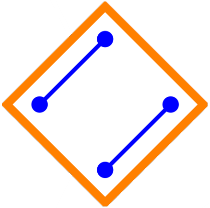
Set partitions



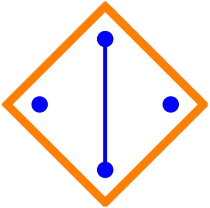
$\{1\}, \{2, 3, 4\}$



$\{1, 2\}, \{3, 4\}$



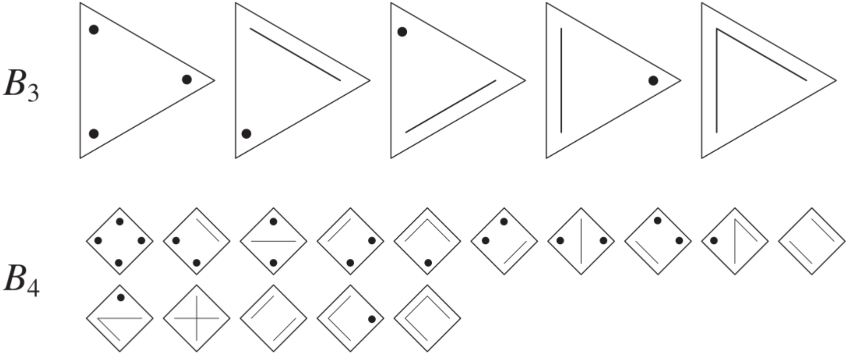
$\{1, 4\}, \{2, 3\}$



$\{1, 3\}, \{2\}, \{4\}$

- Partition of $\{1, ..., n\}$ = disjoint grouping of elements from $\{1, ..., n\}$
- Diagrammatically, connect n dots on a line into groups

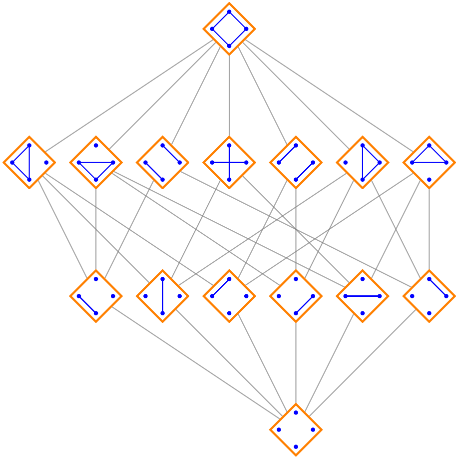
Counting set partitions



- Many counting problems are related to set partitions
- The number of set partitions is given by the Bell numbers B_n
- Theorem The generating function is

$$\sum_{j=0}^{\infty} B_n \frac{z^j}{j!} = \exp(\exp(z-1)) = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \ldots$$

The lattice of set partitions



- Refinement gives set partitions a partial order
- Theorem This partial order has greatest lower and least upper bounds
- This thus forms a lattice

Enter, the theorems

The set partition lattices satisfy several properties:

- It is a geometric lattice
- Any finite sublattice of a free lattice can be embedded into a set partition lattice

  Almost Cayley's theorem for lattices

Cayley's theorem

From Wikipedia, the free encyclopedia

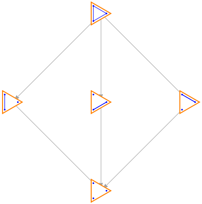
For the number of labeled trees in graph theory, see Cayley's formula.

In group theory, **Cayley's theorem**, named in honour of Arthur Cayley, states that every group G is isomorphic to a subgroup of a symmetric group.

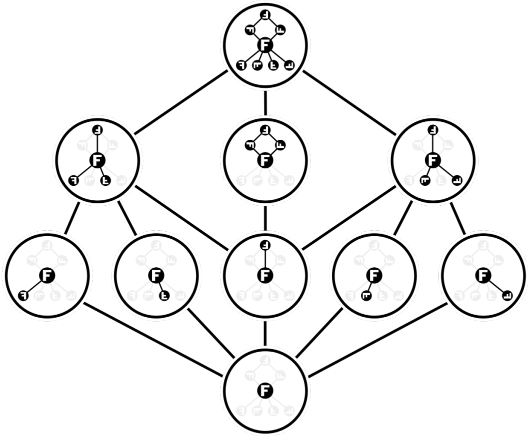
- Properties that hold for all set partition lattices hold for all finite lattices

  "Set partition lattices do not satisfy any nontrivial identity"

- The maximal chain is of length n for $X = \{1, ..., n\}$



Cayley's theorem for lattices



- Subgroups of groups form a lattice by inclusion
- Any lattice can be embedded into a subgroups lattice
  Cayley's theorem for lattices

Thank you for your attention!

I hope that was of some help.