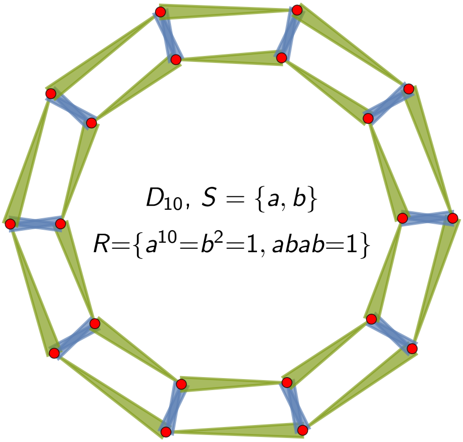
What are...hyperbolic groups?

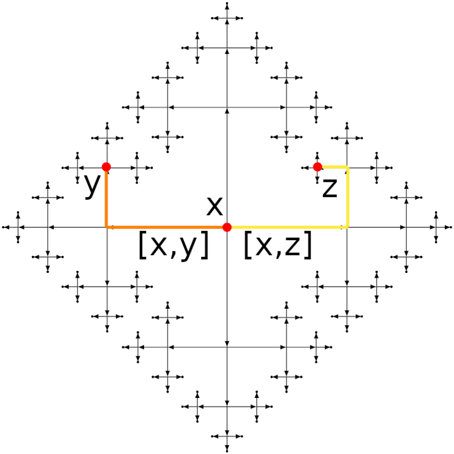
Or: From groups to Cantor sets

Cayley graph



- Every group G generated by S has an associated Cayley graph Γ
- Vertices of Γ = elements of G, edges of Γ = actions of S
- Path-distance defines a metric d on Γ by $d(\text{neighbors in } \Gamma) = 1$

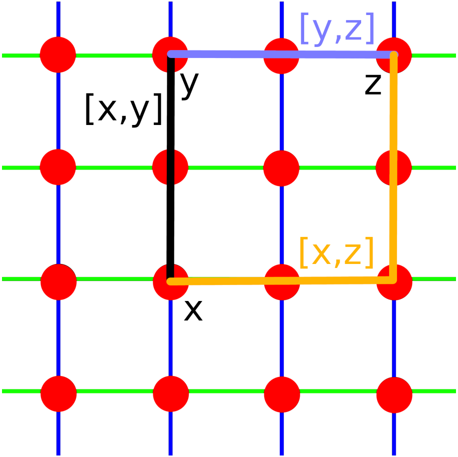
Thin triangles



- The Cayley graph Γ of the free group F_2 with $S = \{a, b\}$ is a tree as above
- Geodesic triangles (x, y, z) in Γ are thin:

$$\exists \delta \geq 0 : \forall w \in [x, y] \text{ we have } d(w, [x, z] \cup [y, z]) \leq \delta$$

Thick triangles



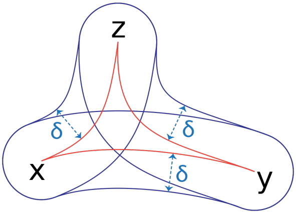
- The Cayley graph Γ of $\mathbb{Z}^2, S = \{(1,0),(0,1)\}$ is a grid as above
- Geodesic triangles (x, y, z) in Γ are not thin:

$$\nexists \delta \geq 0 : \forall w \in [x,y] \text{ we have } d(w,[x,z] \cup [y,z]) \leq \delta$$

Enter, the definition/theorem

A finitely presented group G is hyperbolic if all geodesic triangles (x, y, z) in Γ are thin

$$(*)\exists\delta \geq 0 : \forall w \in [x, y] \text{ we have } d(w, [x, z] \cup [y, z]) \leq \delta$$



Hyperbolic group

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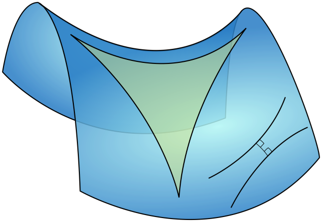
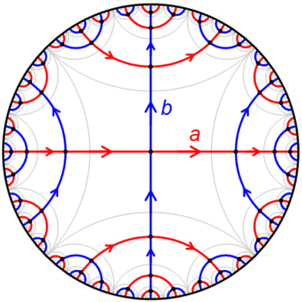


Hyperbolic space

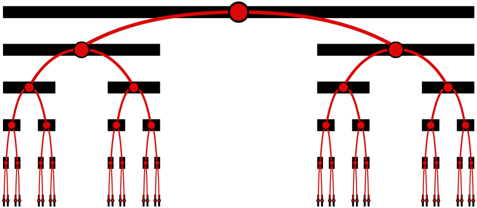
- (*) is called the Rips condition, which is equivalent to triangles being thin
- This does not depend on Γ Intrinsic to G
- Theorem Almost all groups are hyperbolic (very probably random groups are hyperbolic)

Boundaries!

Γ:



Cantor:



- The points at infinity, the boundary, of Γ forms a compact metrizable space
- Theorem The boundary of most hyperbolic groups are Cantor sets/Menger sponges

Thank you for your attention!

I hope that was of some help.