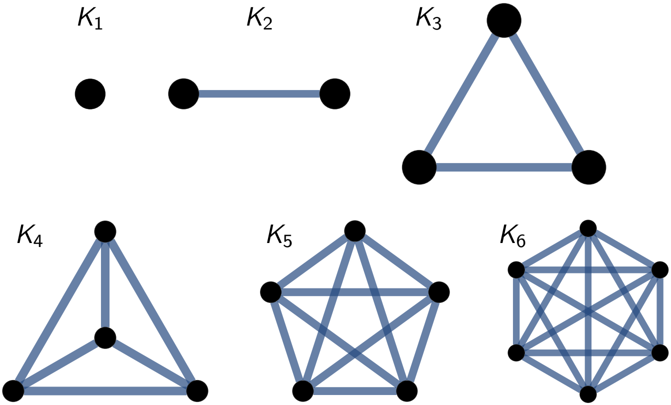
What are...Ramsey numbers?

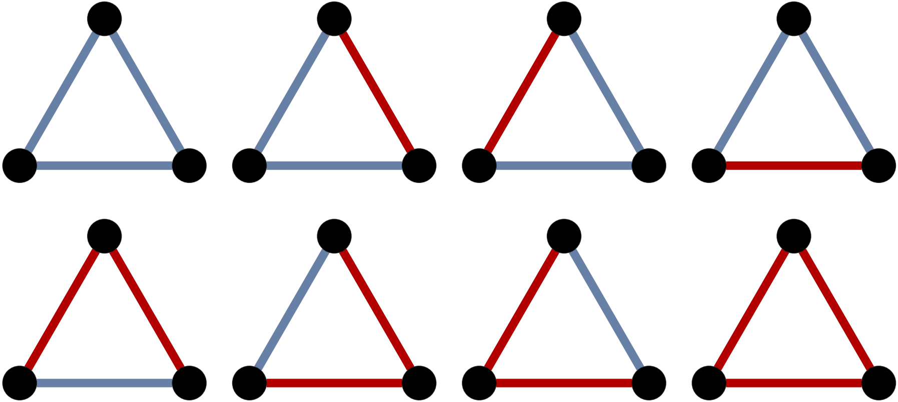
Or: Counting made difficult

Complete graphs K_n and counting



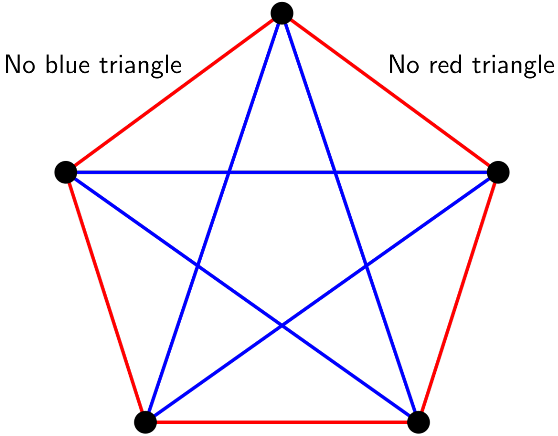
- K_n = graph with n vertices, every pair of vertices is connected by an edge
- No double edges or loops
- K_n is at the heart of many counting problems

Color K_n blue-red



- K_n has $n(n-1)/2$ edges
- There are thus $2^{n(n-1)/2}$ blue-red colorings of the edges

How big must X be to guarantee that Y holds?



- $R(b,r)$ What is the smallest n such that K_n contains a blue K_b or a red K_r for all blue-red colorings of its edges?
- The above says $R(3,3) > 5$ and actually $R(3,3) = 6$

Enter, the theorem

The number $R(b, r) = R(r, b)$ is finite:

$$R(b, r) \leq \binom{b + r - 2}{b - 1}$$

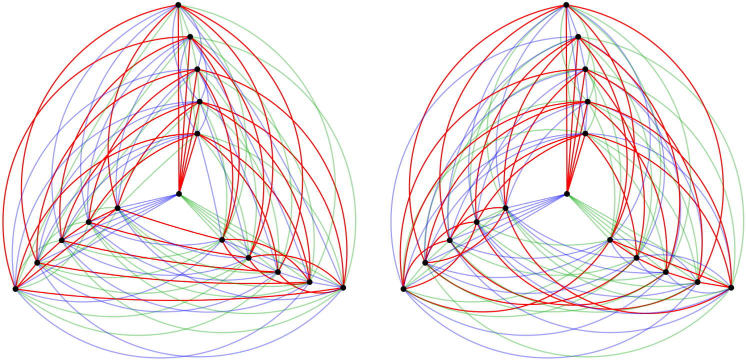
The same works for any finite number of colors: $R(c_1, \dots, c_n) < \infty$

- The numbers are only known (in 2021) for very few values:

$b \backslash r$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8	9	10
3			6	9	14	18	23	28	36	40-42
4				10	25	36-41	49-61	59-84	73-115	92-149
5					43-48	58-87	80-143	101-216	133-316	204-1171
6						102-165	115-298	134-495	183-780	204-1171
7							205-540	219-1031	252-1713	292-2826
8								282-1870	329-3583	343-6090
9									565-6588	581-12677
10										798-23556

- Computing $R(6, 6)$ is probably hopeless

A hopeless counting problem



It is easy to see that $R(3,3,3) \leq 17$, it is hard to see that $R(3,3,3) = 17$:

- Use $R(3,3) = 5$ to see $R(3,3,3) \leq 17$
- K_{16} has $3^{120} \approx 10^{57}$ blue-red-green colorings
- Only 2 have no monochromatic triangle (up to symmetry)

Thank you for your attention!

I hope that was of some help.