What is...the Gessel–Viennot lemma?

Or: Determinants are graphs?!

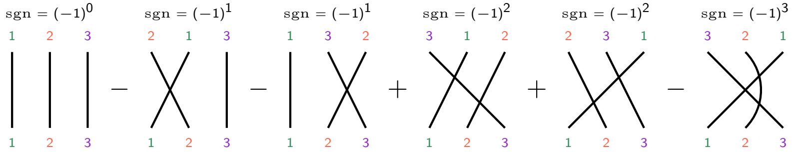
The starting point: a classic!

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) m_{1,\sigma_1} \cdot \ldots \cdot m_{n,\sigma_n}$$

3x3 Example

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

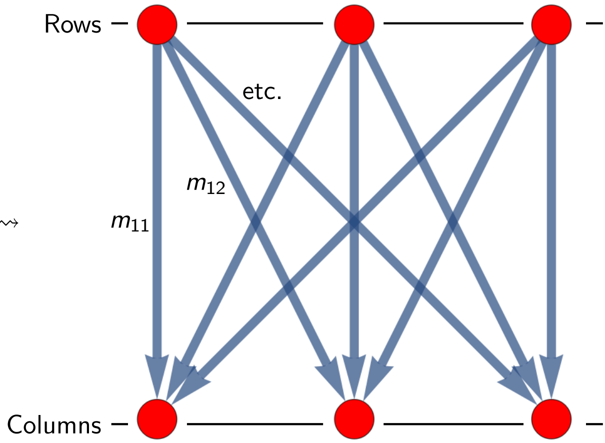
$\det(M) =$



$$= m_{11}m_{22}m_{33} - m_{12}m_{21}m_{33} - m_{11}m_{23}m_{32} + m_{12}m_{23}m_{31} + m_{13}m_{21}m_{32} - m_{13}m_{22}m_{31}$$

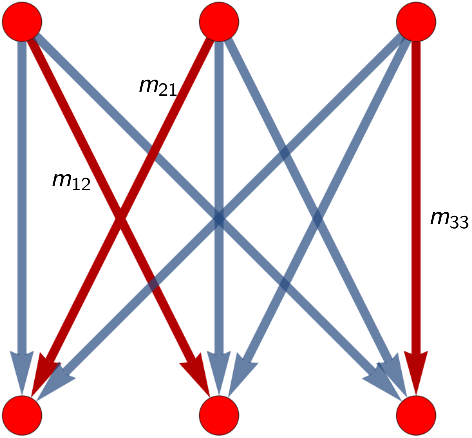
Reinterpretation via graphs

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \leftrightsquigarrow$$



- Matrix M $\leftrightsquigarrow$ weighted directed bipartite graph
- $\det(M)$ $\leftrightsquigarrow$ weighted (signed) sum over vertex-disjoint path systems (vdps)

Determinants and paths



$m_{12} m_{21} m_{33} \leftrightsquigarrow$

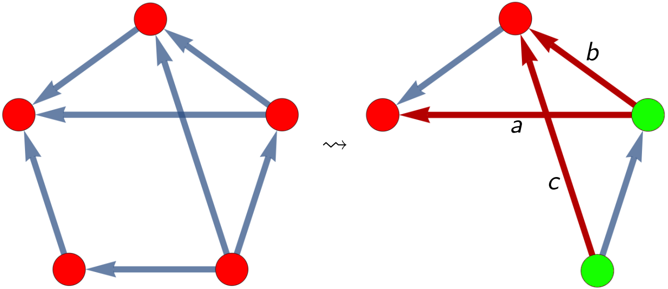
- A vdps P_σ, *e.g.* $P_\sigma = \{1 \to 2', 2 \to 1', 3 \to 3'\}$
- Weights $w(P_\sigma)$, *e.g.* $w(P_\sigma) = m_{12} m_{21} m_{33}$
- Reinterpretation $\det(M) = \sum_{vdps} \mathrm{sgn}(P) w(P)$

Enter, the theorem/lemma

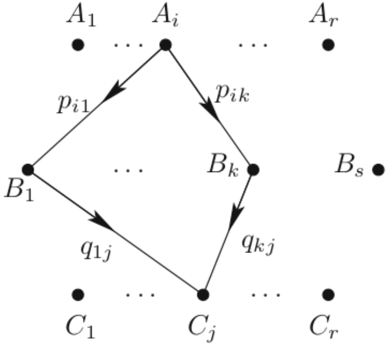
Given a finite weighted acyclic-directed graph, A and B two n-sets of vertices, and M the path matrix from A to B. Then:

$$\det(M) = \sum_{vdps} \operatorname{sgn}(P) w(P)$$

- This generalizes the determinant formula to a huge class of graphs
- Example A=green, B=red, and



$$M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad \det(M) = ac = ac - b0$$



Cauchy–Binet P an rxs matrix, Q an sxr matrix, $r \leq s$, then

$$\det(PQ) = \sum_{Z} \det(P_Z) \det(Q_Z)$$

where P_Z, Q_Z = rxr with column-set Z respectively row-set Z

Proof? The picture above

Thank you for your attention!

I hope that was of some help.