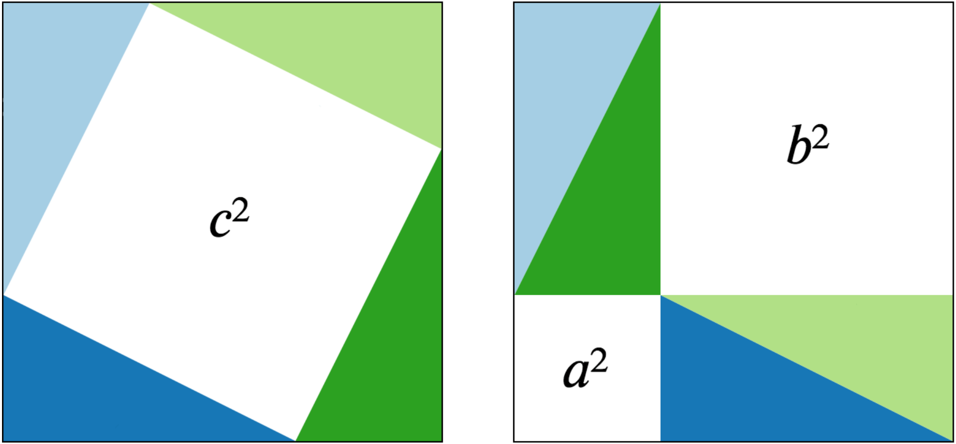
What is...reverse mathematics?

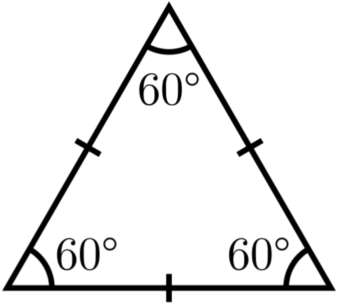
Or: Thinking backwards

The inputs of $a^2 + b^2 = c^2$



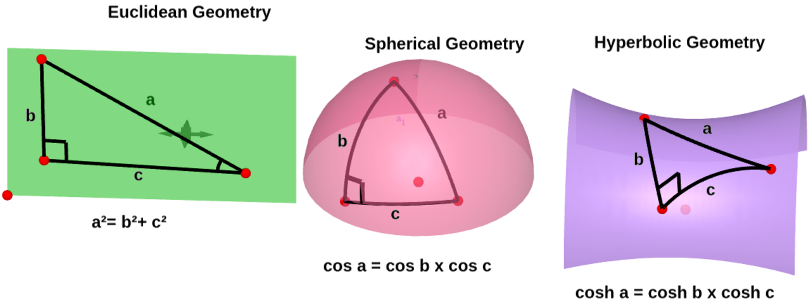
What inputs does this proof need? Do all proofs need the same inputs?

Euclid's axioms



- 1 Between two points there is a line
- 2 Finite lines can be extended to infinite lines
- 3 Circles can be drawn
- 4 All right angles are congruent
- 5 The sum of angles in a triangle is 180 degrees
- I Some implicit "rules of common sense"

Ups, Pythagoras theorem is equivalent to $\boxed{5}$



- 1+2+3+4+ 5 +I ⇒ Pythagoras theorem The proof from before
- 1+2+3+4+Pythagoras theorem+I ⇒ 5 A bit tricky but true
- 1+2+3+4+ 5 +I makes sense Euclidean geometry
- 1+2+3+4+"Not 5 "+I makes sense Spherical/hyperbolic geometry

We just discovered reverse mathematics!

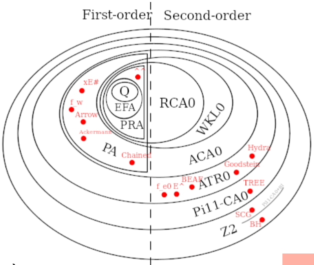
Enter, the theorem/philosophy!

Reverse mathematics seeks right axioms to prove theorems already known

When the theorem is proved from the right axioms,

the axioms can be proved from the theorem – Friedman

Reverse mathematics is part of logic, using first- ("the axioms of natural numbers"), second- ("the axioms of real numbers") or higher-order arithmetic ("beyond")



- Peano's arithmetic (PA) is first-order logic and too weak to formulate $\mathbb{R}$
- Second-order arithmetic (Z_2) is strong enough to formulate $\mathbb{R}$

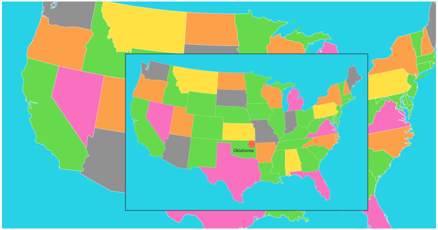
Examples of “equal” theorems

RCA_0 proves Intermediate Value Theorem.

WKL_0 proves Sequential Heine-Borel Theorem
$\Leftrightarrow$ Uniform Continuity Theorem
$\Leftrightarrow$ Extreme Value Theorem
$\Leftrightarrow$ Riemann Integrability of Continuous Functions
$\Leftrightarrow$ Brouwer Fixed Point Theorem
$\Leftrightarrow$ Jordan Curve Theorem
(Equivalences provable in RCA_0).

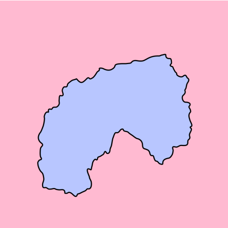
ACA_0 proves Kőnig Infinity Lemma
$\Leftrightarrow$ Sequential Bolzano-Weierstrass Theorem
$\Leftrightarrow$ Sequential Least Upper Bound Property
$\Leftrightarrow$ Cauchy Convergence Criterion
(Equivalences provable in RCA_0).

Brouwer



“equals”

Jordan



Thank you for your attention!

I hope that was of some help.