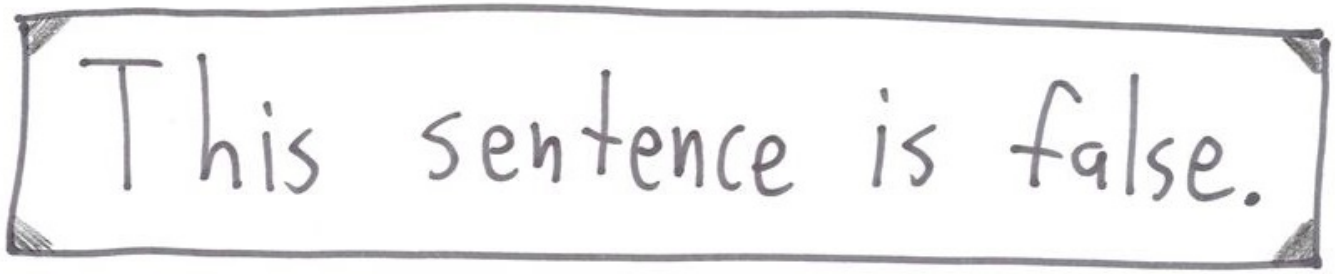
What is...the axiom of choice?

Or: It works even if you don't believe in it

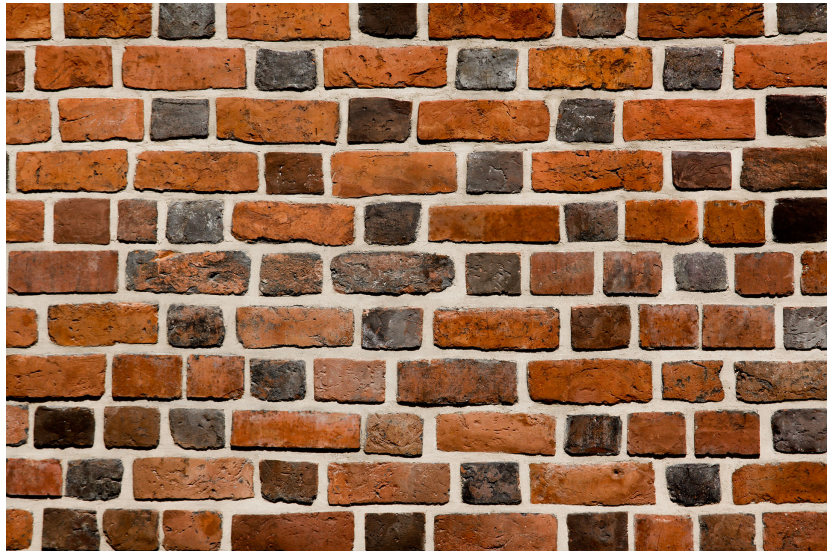
Enter, Russell and self-reference



For $R = \{x \mid x \in x\} : R \in R \Leftrightarrow R \notin R$

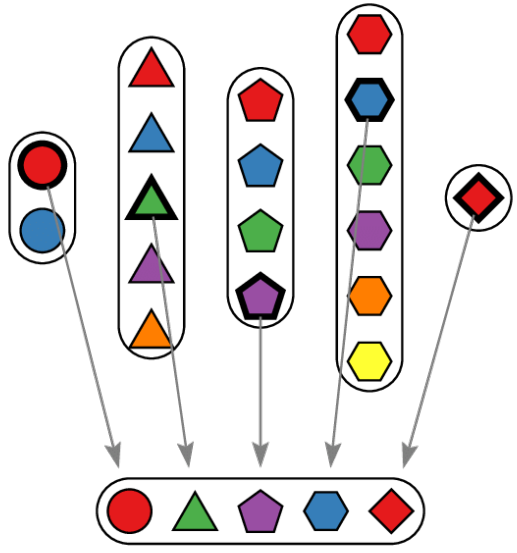
This triggered a need for a solid foundation of mathematics

Enter, Zermelo–Fraenkel's (ZF) bricks



- There exists a(n infinite) set — No empty theory
- $X = Y \Leftrightarrow$ they have the same elements — Setting the stage
- X, Y are set $+ *$ is a good operation $\Rightarrow X * Y$ is a set — Walls from bricks
- There are no downward infinite membership chains — "Induction"
- Every family of nonempty sets has a choice function — Axiom of choice (AC)

Enter, (AC)



No problem for finite sets, but the axiom ensures the choice function for infinite sets

- Family of shoe pairs, "take the left shoe" is a choice function
  No axiom needed
- Family of sock pairs a choice function exists by (AC)
  Socks are indistinguishable

Enter, the theorems

Of course not, but I am told it works even if you don't believe in it – Niels Bohr

(Answer to "Do you believe a horseshoe hanging over your door brings you luck?")

- Gödel $\neg$(AC) is not a theorem of ZF
- Cohan (AC) is not a theorem of ZF
- $\Rightarrow$ (AC) is logically independent of ZF

Its not the preferred choice, but...

Disasters without choice

- Several versions of "finite", all equivalent in ZF+(AC), but not so in ZF
- Without (AC) vector spaces may have no bases
- Without (AC) graphs with all finite subgraphs being 2-colerable might not be
- Many more...

Disasters with choice

- With (AC) there are many non–continuous solutions to $f(x+y) = f(x) + f(y)$
- With (AC) there are many sets without volume
- With (AC) $\mathbb{R} \cong \mathbb{R} \oplus \mathbb{Q}$ as $\mathbb{Q}$-vector spaces
- Many more...

Disasters either way

- (AC) $\Rightarrow$ existence of winning strategies for certain deterministic games
- (AC) $\Rightarrow$ non-existence of winning strategies for certain deterministic games
- Many more...

Thank you for your attention!

I hope that was of some help.