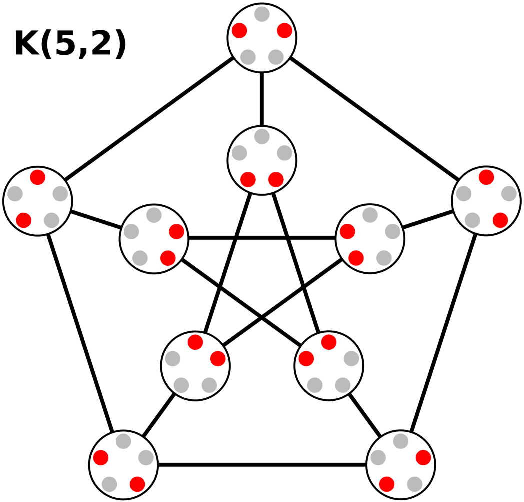
What is...Kneser's conjecture?

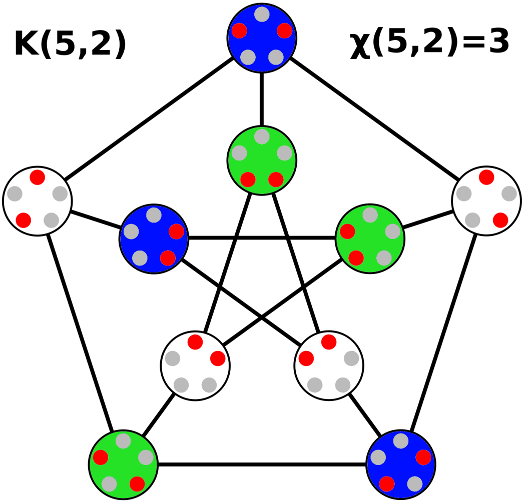
Or: Coloring and topology

Sets and (Kneser) graphs



- $K(n,k)$ – vertices k-element subsets of $\{1,...,n\}$, edges between $A \cap B = \emptyset$
- $K(n,1)$ is the complete graph with n vertices
- $K(n,k)$ is edge-less for $n < 2k$ so we always assume $n \geq 2k$

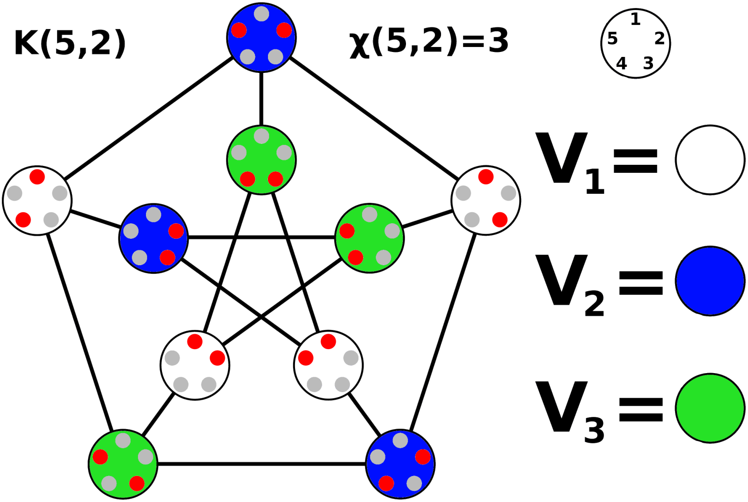
Coloring $K(n,k)$



Problem. What is the smallest $\chi(n,k)$ such that the vertices of $K(n,k)$ can be partitioned into $V_1 \dot\cup \ldots \dot\cup V_\chi$ of intersecting family of k-sets V_i?

Kneser's conjecture (Aufgabe 360). $\chi(2k+d,k) = d+2$

Proof that $\chi(2k+d,k) \leq d+2$



- For $i = 1, \ldots, d+1$ take V_i = k-sets with minimal element i
- For $i = d+2$ combine all remaining sets to V_{d+2}

Enter, the theorem

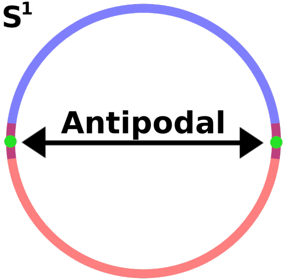
Kneser's conjecture holds

The first proof 23 years after Kneser stated the conjecture used the following version of the Borsuk–Ulam theorem

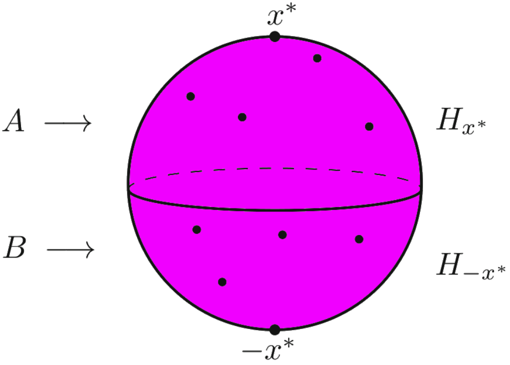
Theorem (Lyusternik–Shnirel'man). If the sphere S^{d+1} is covered by $d+2$ sets

$$S^{d+1} = \bigcup_{i=1}^{d+1} U_i \cup C$$

with U_i open, then one of the U_i or C contains an antipodal pair $x^*, -x^*$



Proof that $\chi(2k+d,k) \geq d+2$



- Take $2k+d$ points in general position on S^{d+1}
- Assume that we have partition $V_1 \dot\cup \ldots \dot\cup V_{d+1}$
- Define open sets $O_i = \{x \in S^{d+1} \mid$ the open hemisphere H_x with pole x contains a k-set from $V_i\}$
- Borsuk–Ulam theorem $\Rightarrow$ some O_i contains antipodal points x, x^*
- Thus, we get two sets A, B in V_i which are disjoint

Thank you for your attention!

I hope that was of some help.