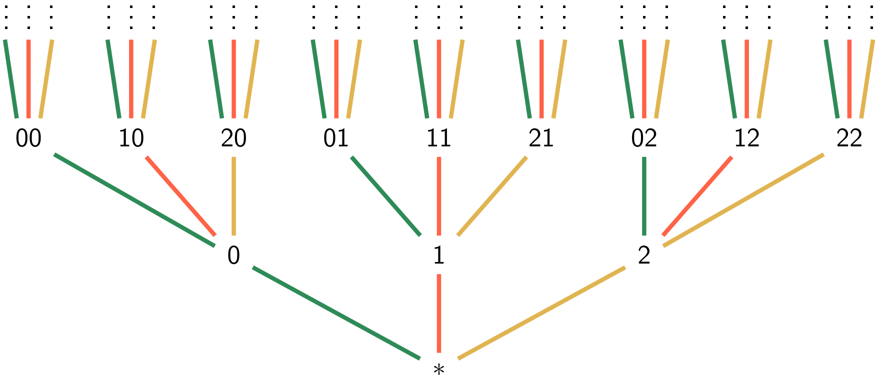
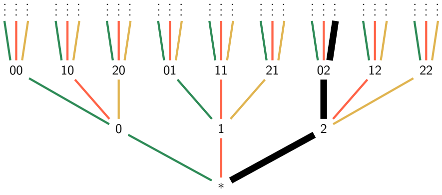
What are...p-adic integers?

Or: Climbing infinite trees

p-adic integers $\mathbb{Z}_p$ are walks from the root to a leaf



$\dots 202_3$ ⇝



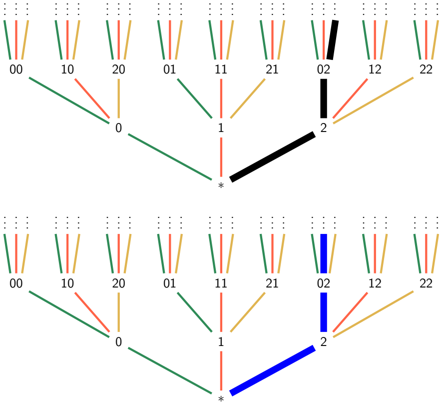
Distance of walks in a tree

We have a tree metric $d(a,b) = 1/p^k$ with

k = distance of the first branching point to the root

$\ldots 202_3 \leftrightsquigarrow$

$\ldots 102_3 \leftrightsquigarrow$



$$d(\ldots 202_3, \ldots 102_3) = 1/3^2$$

Metric? Check! Addition 7-adically? Check!

Ansatz: 7-adic numbers are $(\dots a_2 a_1 a_0)_7$ for $a_k \in \{0, \dots, 6 = 7 - 1\}$

Addition of $\dots 251413_7$ and $\dots 121102_7$

Carry	...	1					
	...	2	5	1	4	1	3
+	...	1	2	1	1	0	2
	...	4	$7 = 0$	2	5	1	5

What about $n \in \mathbb{Z}$? Apply subtraction from elementary arithmetic:

Carry	...	1	1	1	1	1	
	...	0	0	0	0	0	0
−	...	0	0	0	0	0	1
	...	6	6	6	6	6	6

$-1 = \dots 666666_7$ in analogy to $1 = 0.999999\dots_{10}$

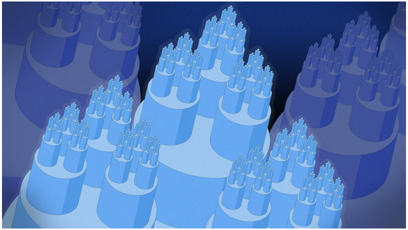
Enter, the theorem

p-adic integers $\mathbb{Z}_p$ and numbers $\mathbb{Q}_p$ exist via the equivalent definitions

(a) $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Q}_p$ is its field of fractions Algebra

(b) $\mathbb{Q}_p = \frac{(\text{Cauchy sequences in } \mathbb{Q} \text{ wrt } d)}{(\text{Nil sequences in } \mathbb{Q} \text{ wrt } d)}$ and $\mathbb{Z}_p$ is its ring of integers Analysis

$\mathbb{R} = \frac{(\text{Cauchy sequences in } \mathbb{Q} \text{ wrt } d)}{(\text{Nil sequences in } \mathbb{Q} \text{ wrt } d)}$ for the standard metric Analogy



Theorem (local-global). Let $f \in \mathbb{Q}[X_1, ..., X_n]$ be nice

(a) If $f = 0$ holds in $\mathbb{Q}$, then it holds in $\mathbb{R}$ and $\mathbb{Q}_p$ for all p global→local

(b) If $f = 0$ holds in $\mathbb{R}$ and $\mathbb{Q}_p$ for all p, then it holds in $\mathbb{Q}$ local→global

Newton and $\sqrt{2}$ in p-adics

Solution by Newton's method of $x^2 - 2 = 0$

Suggestions: (p, x_0) = (7, 3), (7, 4), (17, 6), (17, 11), (23, 5), (23, 18), (31, 8), (31,23)

n	x_n	x_n as fraction	x_n as p−adic	check: x_n^2 as p−adic
0	3	3	3_7	12_7
1	1.833-33	$\frac{11}{6}$	$...1111111111111113.0_7$	$...3206543206543210 2.0_7$
2	1.462-12	$\frac{193}{132}$	$...3306211352330621 3.0_7$	$...1515640034331000 2.0_7$
3	1.415	$\frac{72\,097}{50\,952}$	$...0162352532121621 3.0_7$	$...0601033510000000 2.0_7$
4	1.414-21	$\frac{10\,390\,190\,017}{7\,346\,972\,688}$	$...0201126642121621 3.0_7$	$...1000000000000000 2.0_7$

$\sqrt{2} \approx ...216213.0_7$ 7-adically

Thank you for your attention!

I hope that was of some help.